2. Consider a model in which electrons have a number density \( n_e \) and are in a damped harmonic oscillator, such that their displacement \( x \) in the presence of an electric field will be governed by \( m \ddot{x} = -eE - m \gamma \dot{x} - m \omega_0^2 x \). Assuming the positions and electric field are both proportional to \( e^{-i \omega t} \), find the relationship between \( E \) and \( x \). Then find the polarization \( P = -n_e e x \), and the complex permittivity \( \varepsilon \). As a check, make sure you get the same answer as we did for a collisionless plasma \( (\gamma = \omega_0 = 0) \).

Rewriting \( x(t) = x e^{-i \omega t} \) and \( E(t) = E e^{-i \omega t} \), we see that all time derivatives just become factors of \( -i \omega \), so we have

\[-m \omega^2 x = -eE + i m \gamma \omega x - m \omega_0^2 x\]

Solving for \( x \), we have

\[m \omega_0^2 x - m \omega^2 x - i m \gamma \omega x = \frac{-eE}{m (\omega_0^2 - \omega^2 - i m \gamma \omega)}\]

The polarization will just be \( P = -n e x \), and therefore

\[D = \varepsilon_0 E + P = \varepsilon_0 E + \frac{n e^2 E}{m (\omega_0^2 - \omega^2 - i m \gamma \omega)}\]

Since \( D = \varepsilon E \), the permittivity is therefore

\[\varepsilon = \varepsilon_0 + \frac{n e^2}{m (\omega_0^2 - \omega^2 - i m \gamma \omega)}\]

If we compare this with the formula for a plasma, we have

\[\varepsilon = \varepsilon_0 + \frac{i \sigma}{\omega} = \varepsilon_0 - \frac{n e^2}{m \omega_0^2}\]

This is clearly the same equation if we let \( \gamma = \omega_0 = 0 \).
3. What is the real part of the permittivity of a material if

\[ \text{Im} \left[ \varepsilon(\omega) \right] = \begin{cases} a(\omega_0^2 - \omega^2) & \text{for } \omega < \omega_0, \\ 0 & \text{for } \omega > \omega_0. \end{cases} \]

It should be noted that you will not have to use the principal part when attempting to find \( \text{Re} \left[ \varepsilon(\omega) \right] \) if \( \omega > \omega_0 \).

We simply use the Kramers-Kronig relationship

\[
\text{Re} \left[ \varepsilon(\omega) \right] = \varepsilon_0 + \frac{2a}{\pi} \int_0^{\omega} \frac{\omega' \text{Im} \left[ \varepsilon(\omega') \right]}{\omega'^2 - \omega^2} d\omega' = \varepsilon_0 + \frac{2a}{\pi} \int_0^{\omega} \frac{\omega'^2}{\omega'^2 - \omega^2} d\omega' \\
= \varepsilon_0 + \frac{2a}{\pi} \int_0^{\omega} \left[-\omega'^2 + (\omega_0^2 - \omega^2) \left(1 - \frac{1}{2} \frac{\omega}{\omega' + \omega} + \frac{1}{2} \frac{\omega}{\omega' - \omega} \right)\right] d\omega' \\
= \varepsilon_0 + \frac{2a}{\pi} \left\{ -\frac{1}{2} \omega'^3 + (\omega_0^2 - \omega^2) \left[\omega' - \frac{1}{2} \omega \ln(\omega' + \omega) + \frac{1}{2} \omega \ln(\omega' - \omega) \right] \right\}_{0}^{\omega}.
\]

Only the final term requires any care in evaluating the limit, as we must avoid the pole at \( \omega' = \omega \). When \( \omega > \omega_0 \), there is no pole, and this last term just yields \( \ln(\omega - \omega_0) - \ln(\omega_0) \), but if \( \omega < \omega_0 \), then

\[
P(\ln|\omega' - \omega|)_{0}^{\omega} = \lim_{\delta \to 0^+} \left[ (\ln|\omega' - \omega|)_{0}^{\omega - \delta} + (\ln|\omega' - \omega|)_{\omega + \delta}^{\omega} \right] \\
= \lim_{\delta \to 0^+} \left[ -\delta - \ln|\omega_0 - \omega| + \ln|\omega_0 - \omega| - \ln|\delta| \right] = \ln \left( \frac{\omega_0 - \omega}{\omega_0} \right).
\]

The last expression works in both cases. So we have

\[
\text{Re} \left[ \varepsilon(\omega) \right] = \varepsilon_0 + \frac{2a}{\pi} \left\{ \frac{2}{3} \omega_0^3 - \omega_0 \omega^2 + \frac{1}{2} \omega \left( \omega_0^2 - \omega^2 \right) \left[ \ln \left( \frac{\omega - \omega_0}{\omega} \right) - \ln \left( \frac{\omega + \omega_0}{\omega} \right) \right] \right\} \\
= \varepsilon_0 + \frac{2a}{\pi} \left\{ \frac{2}{3} \omega_0^3 - \omega_0 \omega^2 + \frac{1}{2} \omega \left( \omega_0^2 - \omega^2 \right) \ln \left( \frac{\omega - \omega_0}{\omega + \omega_0} \right) \right\}.
\]