This test consists of five parts. Please note that in parts II through V, you can skip one question of those offered.

**Part I: Multiple Choice (mixed new and review questions) [50 points]**

For each question, choose the best answer (2 points each)

1. When studying the infinite square well, with allowed region $0 < x < L$, we used functions like $\sin(kx)$ but rejected functions like $\cos(kx)$. Why?
   A) Cosine doesn’t satisfy Schrödinger’s equation
   B) We need it to fall to zero at infinity, and cosine doesn’t do that
   C) It is impossible to normalize cosine, unlike sine
   D) The two are equivalent (just shifted by a phase), so there was no reason to study both
   E) **The function must be continuous at $x = 0$, and sine satisfies this but cosine does not**

2. The particle that is exchanged between nucleons in the nucleus of an atom is called the
   A) Pion  B) Photon  C) Gluon  D) Neutrino  E) Graviton

3. When the Sun dies, what sort of star will it be?
   A) Main sequence  B) Black Hole  C) Neutron star  **D) White dwarf**  E) None of these

4. In general relativity, the source of gravity in Einstein’s equations is called the
   A) Mass  B) Energy  C) Momentum  D) Connection  **E) Stress-energy tensor**

5. What happens as you approach a large mass, according to general relativity?
   A) Mass increases
   B) Mass decreases
   **C) Time slows down**
   D) Time speeds up
   E) None of the above

6. In order to knock an electron from a metal, you must bombard it with light which has a
certain minimum
   A) Intensity  B) Frequency  C) Wavelength  D) Number of photons  E) None of these

7. As mass increase, the radius of a main sequence star ______ and a white dwarf _______
A) Increases, increases  
B) Decreases, decreases  
C) Increases, decreases  
D) Decreases, increases  
E) Stays about the same, stays about the same

8. To calculate the degeneracy pressure of a white dwarf, we need to know the number density of
   A) Electrons  B) Protons  C) Neutrons  D) Nuclei  E) Photons

9. If a wave impacts a step function whose height $V_0$ is larger than the energy $E$, what happens to the wave in the region with the high potential?
   A) The wave continues as a plane wave with a very small amplitude into this region  
   B) The wave continues as a large plane wave in this region, but with a different wavelength  
   C) The wave is exponentially damped in this region, quickly falling to zero  
   D) The wave alternately moves to the right and to the left in this region, but on the average goes nowhere  
   E) There is no wave at all in this region

10. The fourth component of momentum is
    A) Mass  B) Motion  C) Energy  D) Time  E) Position

11. Rutherford scattered alpha particles off of nuclei, and thereby was able to measure
    A) The charge of the alpha particle  
    B) The charge of the nucleus  
    C) The mass of the nucleus  
    D) The size of the nucleus  
    E) The size of the atom

12. The expectation value of an operator tells you
    A) The single value you will get if you measure something each time  
    B) The most likely value (mode) you will get if you measure it multiple times  
    C) The average value you will get if you measured a system multiple times  
    D) The value it would have if the object were a classical (not quantum) object  
    E) The value that should be assumed for the operator if you need to substitute it into another equation

13. Gravitational waves were first discovered in 2015, and announced in 2016. Before they were discovered, what sign did we have that they already existed?
    A) The distance to the Sun was oscillating as predicted by the waves  
    B) The frequencies of atoms were changing as waves passed them  
    C) The orbit of Mercury was decaying from the gravitational waves it emitted  
    D) The Sun was vibrating from passing gravitational waves  
    E) The period of pairs of neutron stars orbiting each other was gradually decaying

14. According to the principal of equivalence, which of the following is equivalent to being in a gravitational field?
A) Being in an accelerated reference frame
B) Feeling a uniform force
C) Having constant curvature
D) Being on a rotating object
E) Feeling a magnetic field

15. Which nuclei are likely to undergo $\beta^-$ decay?
   A) Those that have very large mass
   B) Those that have extra internal energy
   C) Those that have large numbers of electrons in the nucleus
   D) Those with an excess of protons
   E) Those with an excess of neutrons

16. If the wave function is given by $\psi(x)$, then the probability density of finding the particle at $x$ is given by
   A) $\psi$
   B) $\psi^*$
   C) $\psi + \psi^*$
   D) $\psi^2$
   E) $|\psi|^2$

17. Which of the following equations is still valid, according to special relativity?
   A) $W = Fd$
   B) $p = mv$
   C) $E = \frac{1}{2}mv^2$
   D) $F = ma$
   E) None of these

18. In quantum mechanics, momentum does not have a specific real value, instead it is considered a
   A) Wave function
   B) Operator
   C) Complex number
   D) Hamiltonian
   E) Projector

19. Which argument suggests that in special relativity, there is no such thing as rigid objects?
   A) All objects must be held together by electrons, which travel at a finite speed
   B) A rigid objects would have both ends stopping simultaneously, but simultaneity is poorly defined in special relativity
   C) At high speeds, objects get shorter, which proves they are not rigid
   D) Objects are always held together by photons, which travel only at the speed of light
   E) The fact that moving objects have different energies proves they are being compressed like a spring

20. In the twin paradox, when someone moves fast away from the Earth and then returns, while the other one remains on Earth, what determines which one ends up younger?
   A) It is the one that is actually moving, as compared to the background ether
   B) It is the one that is actually at rest, as compared to the background ether
   C) It is the one that is accelerating
   D) Is the one that is not accelerating
   E) It depends on which observer you ask

21. Iron (Fe) is element 26. What is the approximate mass of $^{56}\text{Fe}$?
   A) 26 u
   B) 30 u
   C) 82 u
   D) 56 u
   E) None of these
22. Which of the following spectral class of stars would have the highest surface temperature?  
A) A  B) B  C) K  D) M  E) F

23. There is a maximum mass for a neutron star. If you added mass to a neutron star at this maximum, what would happen?  
A) It would undergo rapid expansion from the added mass, and turn into a white dwarf  
B) Initial collapse would trigger runaway nuclear explosions, which would cause a supernova  
C) It would start to rotate and become a pulsar  
D) It would collapse and become a black hole  
E) None of the above

24. For which types of radioactive decay is a neutrino or anti-neutrino also emitted?  
A) $\alpha$ decay (only)  
B) $\beta$ decay (only)  
C) $\gamma$ decay (only)  
D) $\alpha$ and $\beta$ decay, but not $\gamma$ decay  
E) $\alpha$, $\beta$, and $\gamma$ decay

25. The energy of a single photon of light is given by  
A) $hf$  
B) $h/f$  
C) $h\lambda$  
D) $h/\lambda$  
E) $\lambda/h$
Part II: Short answer (review material) [20 points]

Choose two of the following three questions and give a short answer (1-3 sentences) (10 points each).

26. A muon lasts typically \( \tau = 2 \times 10^{-6} \text{ s} \), and since they always travel slower than the speed of light, they typically travel no more than \( c\tau = 600 \text{ m} \) before decaying. Nonetheless, muons that are produced high in the atmosphere, many km up, are seen at Earth’s surface. How is this possible? You should provide at least one equation.

Although this is the mean lifetime as measured in the muon’s own frame, there is a small but significant probability that they may live longer, perhaps as much as ten times longer, than the mean. Far more important, however, is the fact that these muons can be moving at nearly the speed of light, and therefore experience time dilation, such that the mean lifetime as measured by an observer on Earth would be \( \Delta t = \gamma \tau \), where \( \gamma = 1/\sqrt{1-v^2/c^2} \). Hence, for example, a muon traveling at 99.995% of the speed of light would last a hundred times longer, or 0.2 ms, and at this speed would go 60 km, before decaying.

27. The harmonic oscillator has potential energy \( \frac{1}{2}kx^2 \), which has a minimum value at \( x = 0 \), where the energy is zero. If a particle is at rest, it will have no kinetic energy. Hence the minimum energy for the harmonic oscillator is zero. Explain what, if anything, is wrong with this reasoning. Your answer should have one equation or inequality.

According to one version of the uncertainty principle, the momentum and position cannot be simultaneously specified, and the corresponding uncertainties are related by \( (\Delta x)(\Delta p) \geq \frac{1}{2} \hbar \). To minimize the potential energy, one would want to specify \( x = 0 \), but this requires making \( \Delta x \) very small, which would force \( \Delta p \) to be large, which means we cannot specify the velocity with high precision. Indeed, this suggests a high velocity and hence a large energy. Setting the energy to zero would require specifying both \( x \) and \( p \), violating the uncertainty principle.

28. Suppose a particle of energy \( E \) impacts a barrier of potential \( V_0 > E \). Describe qualitatively what happens. No equation is expected.

Classically, the particle would be completely absorbed as soon as it reaches the initial step up, but quantum mechanically, the wave continues into the classically forbidden region, although as a rapidly falling exponential. However, since the barrier has only finite width, it will still be non-zero when it gets to the far side of the barrier, at which point it continues as a plane wave, with a substantially reduced amplitude.
Part III: Short answer (new material) [30 points]

Choose three of the following four questions and give a short answer (1-3 sentences) (10 points each).

29. Explain qualitatively why light nuclei, like $^{16}\text{O}$, seem to be composed of about 50% protons. Then explain why for heavy nuclei, like $^{200}\text{Hg}$, the balance is about 40% protons.

Nucleons tend to go into the lowest energy states, and since the neutron and proton have nearly the same mass, and nearly identical strong interactions, it is most energetically favorable to choose to make half of them protons and half neutrons, so as to fill up the lowest energy states possible. However, as the mass increases and the protons become more prevalent, the long-range electromagnetic repulsion

30. Explain qualitatively what degeneracy pressure is. What particle contributes to degeneracy pressure in white dwarfs and neutron stars?

Degeneracy pressure is pressure caused by the Pauli exclusion principle. Two fermions cannot go into the same quantum state, so if you place a number of fermions into a high density region, many of them will have to go into high-energy states, producing high energy. White dwarfs are supported by electron degeneracy pressure, while neutron stars are supported by neutron degeneracy pressure.

31. What is the primary component(s) of the composition of each of the following types of star: main sequence, white dwarf, and neutron star.

Main sequence stars are composed of primarily hydrogen, followed by helium (specifically, $^1\text{H}$ and $^4\text{He}$). White dwarfs are composed of mostly carbon and oxygen ($^{12}\text{C}$ and $^{16}\text{O}$). Neutron stars are made mostly of neutrons.

32. Explain, in general relativity, what a geodesic is. Under what circumstances does an object follow a geodesic?

A geodesic is the longest proper-time path between two points in spacetime. In general relativity, an object follows a geodesic when there are no forces other than gravity acting on it. In particular, if gravity alone is acting on it, it will follow a geodesic.
Part IV: Calculation (review material) [40 points]

Choose two of the following three questions and perform the indicated calculations (20 points each)

33. A domino normally has dimensions of approximately 2.50 cm × 5.00 cm × 0.700 cm, and a mass of 8.00 g. However, a passing domino is moving so fast that it appears to be square.

(a) Which way is the domino moving? What are the measured dimensions of this moving domino?

Since the domino is compressed in the vertical direction, it must be moving in that direction. It may be moving either up or down. It is compressed only in this direction, so the measured dimensions are 2.50 cm × 2.50 cm × 0.700 cm.

(b) What is the speed of the domino, in m/s?

The observed length and proper length are related by \( L = L_p / \gamma \), so

\[
\gamma = \frac{L_p}{L} = \frac{5.00 \text{ cm}}{2.50 \text{ cm}} = 2.00.
\]

Solving for the velocity, we have therefore

\[
\frac{1}{\sqrt{1 - v^2 / c^2}} = 2.00,
\]

\[
1 - v^2 / c^2 = (0.500)^2 = 0.250,
\]

\[
v / c = \sqrt{1 - 0.250} = \sqrt{0.750} = 0.866,
\]

\[
v = 0.866c = 0.866 \left( 2.998 \times 10^8 \text{ m/s} \right) = 2.60 \times 10^8 \text{ m/s}.
\]

(c) What is the energy of this domino in J? What is its momentum in kg⋅m/s?

We simply use the formulas for energy and momentum to obtain

\[
E = \gamma mc^2 = 2 \left( 0.00800 \text{ kg} \right) \left( 2.998 \times 10^8 \text{ m/s} \right)^2 = 1.44 \times 10^{15} \text{ J},
\]

\[
p = \gamma mu = 2 \left( 0.00800 \text{ kg} \right) \left( 2.60 \times 10^8 \text{ m/s} \right) = 4.15 \times 10^6 \text{ kg⋅m/s}.
\]
34. A boron atom \((Z = 5)\) has only one electron in it, in the \(n = 7\) level of the atom. 
(a) How much energy (in eV) would it take to liberate the electron entirely from the atom? Find the minimum frequency of light that can do so.

The binding energy can be found just by using the formula:

\[
E_\gamma = - \frac{(13.6 \text{ eV})^2}{7^2} = -6.94 \text{ eV}.
\]

It therefore would take 6.94 eV to liberate an electron. The frequency can be found with the help of \(E = hf\), so we find

\[
f = \frac{6.94 \text{ eV}}{4.136 \times 10^{-15} \text{ eV} \cdot \text{s}} = 1.68 \times 10^{15} \text{ s}^{-1} = 1.68 \times 10^{15} \text{ Hz}.
\]

(b) Suppose a photon with frequency \(f = 2.53 \times 10^{15} \text{ Hz}\) were to dislodge this electron from boron. What would be the kinetic energy of the ejected electron?

The energy of the photon would be \(E = hf\), but 6.94 eV of this would go to removing the electron and the balance would go into the kinetic energy of the resulting electron. We therefore have

\[
E_{\text{kin}} = hf - |E_\gamma| = \left(4.136 \times 10^{-15} \text{ eV} \cdot \text{s}\right) \left(2.53 \times 10^{15} \text{ s}^{-1}\right) - 6.94 \text{ eV} = 10.46 \text{ eV} - 6.94 \text{ eV} = 3.53 \text{ eV}.
\]

(c) Suppose instead it falls to the \(n = 4\) level. Find the amount of energy released (in eV). Then find the corresponding wavelength of the light that is emitted.

In this case we need to find the difference in the initial and final energies, which will be

\[
\Delta E = E_\gamma - E_4 = - \frac{(13.6 \text{ eV})^2}{7^2} + \frac{(13.6 \text{ eV})^2}{4^2} = 14.31 \text{ eV}.
\]

The corresponding frequency would be

\[
f = \frac{E}{h} = \frac{14.31 \text{ eV}}{4.136 \times 10^{-15} \text{ eV} \cdot \text{s}} = 3.46 \times 10^{15} \text{ s}^{-1}.
\]

We then find the wavelength using \(c = f \lambda\), which we solve for the wavelength to yield

\[
\lambda = \frac{c}{f} = \frac{2.998 \times 10^8 \text{ m/s}}{3.46 \times 10^{15} \text{ s}^{-1}} = 8.66 \times 10^{-8} \text{ m} = 86.6 \text{ nm}.
\]
35. A particle has wave function \( \psi(x) = \begin{cases} 2\lambda \sqrt{x} e^{-\lambda x} & x \geq 0, \\ 0 & x < 0. \end{cases} \)

(a) Is there any place in the region \( x \geq 0 \) that the particle cannot be?

The only place the particle cannot be is where the wave function vanishes, at \( x = 0 \).

(b) What is the most probable place to find the particle?

For a real wave function, this is the place where there is a local maximum or local minimum of the wave function. This is found by setting the derivative to zero, which yields

\[
0 = \frac{d}{dx} \psi(x) = 2\lambda x^{-1/2} e^{-\lambda x} - 2\lambda x^{1/2} e^{-\lambda x} = \lambda \left( x^{-1/2} - 2x^{1/2} \right) e^{-\lambda x},
\]

\[
x^{-1/2} = 2\lambda x^{1/2},
\]

\[
1 = 2\lambda x,
\]

\[
x = \frac{1}{2\lambda}.
\]

(c) What are the expectation value of the position \( \langle x \rangle \) and position squared \( \langle x^2 \rangle \)? A possibly useful integral appears below.

We calculate each of these using the formula \( \langle \mathcal{O} \rangle = \int_{-\infty}^{\infty} \psi^*(x) \mathcal{O} \psi(x) \, dx \), so we have

\[
\langle x \rangle = \int_{-\infty}^{\infty} \psi^*(x) x \psi(x) \, dx = 4\lambda^2 \int_{0}^{\infty} x \left( \sqrt{x} e^{-\lambda x} \right)^2 \, dx = 4\lambda^2 \int_{0}^{\infty} x^2 e^{-2\lambda x} \, dx = 4\lambda^2 \left( 2\lambda \right)^{-3} \Gamma(3)
\]

\[
= \frac{1}{3} \lambda^{-1} 2 = \lambda^{-1},
\]

\[
\langle x^2 \rangle = \int_{-\infty}^{\infty} \psi^*(x) x^2 \psi(x) \, dx = 4\lambda^2 \int_{0}^{\infty} x^2 \left( \sqrt{x} e^{-\lambda x} \right)^2 \, dx = 4\lambda^2 \int_{0}^{\infty} x^3 e^{-2\lambda x} \, dx = 4\lambda^2 \left( 2\lambda \right)^{-4} \Gamma(4)
\]

\[
= \frac{1}{4} \lambda^{-2} 3! = \frac{3}{4} \lambda^{-2}.
\]

(d) What is the uncertainty \( \Delta x \) in the position?

The uncertainty is given by

\[
\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{3}{2} \lambda^{-2} - \lambda^{-1}} = \lambda^{-1} \sqrt{\frac{3}{2} - 1} = \frac{1}{\lambda \sqrt{2}}.
\]

Possibly useful integral:

\[
\int_{0}^{\infty} x^n e^{-a x} \, dx = a^{-n-1} \Gamma(n+1), \quad \Gamma(n+1) = n!,
\]

\[
\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \sqrt{\pi}, \quad \Gamma\left(\frac{5}{2}\right) = \frac{3}{4} \sqrt{\pi},
\]
Part V: Calculation (new material): [60 points]
Choose three of the following four questions and perform the calculations (20 points each)

36. The isotope $^{238}\text{U}$ decays with a half-life of $t_{238} = 4.468 \text{ Gy}$, which quickly becomes $^{206}\text{Pb}$.
The isotope $^{235}\text{U}$ decays with a half-life of $t_{235} = 0.704 \text{ Gy}$, which quickly becomes $^{207}\text{Pb}$.
(a) What are the decay rates $\lambda_{238}$ and $\lambda_{235}$?

We simply use the formula $\lambda_{1/2} = \ln 2 / t$ to find each of these decay rates:

$$
\lambda_{238} = \frac{\ln 2}{t_{238}} = \frac{0.6931}{4.468 \text{ Gy}} = 0.1551 \text{ Gy}^{-1},
$$

$$
\lambda_{235} = \frac{\ln 2}{t_{235}} = \frac{0.6931}{0.704 \text{ Gy}} = 0.9846 \text{ Gy}^{-1}.
$$

(b) A rock sample, initially believed to contain no lead (Pb), has the indicated number of atoms of each isotope shown in the table at right. How many atoms of $^{238}\text{U}$ and $^{235}\text{U}$ were there originally?

Since the $^{206}\text{Pb}$ came from $^{238}\text{U}$, it makes sense that the sum of these two types was the amount of $^{238}\text{U}$. Similarly, the sum of $^{207}\text{Pb}$ and $^{235}\text{U}$ should be the original $^{235}\text{U}$. So we have

$$
N_0(238) = N\left(^{238}\text{U}\right) + N\left(^{206}\text{Pb}\right) = 4.352 \times 10^{10} + 2.098 \times 10^{10} = 6.450 \times 10^{10},
$$

$$
N_0(235) = N\left(^{235}\text{U}\right) + N\left(^{207}\text{Pb}\right) = 2.940 \times 10^{8} + 3.278 \times 10^{9} = 3.572 \times 10^{9}.
$$

(c) By comparing $^{238}\text{U}$ and $^{206}\text{Pb}$, estimate the age of this rock.

We use the formula $N = N_0 e^{-\lambda t}$ to find the age. First, rearrange this to $e^{\lambda t} = N_0 / N$, or taking the logarithm, $\lambda t = \ln \left(N_0 / N\right)$. We therefore have

$$
\lambda t = \ln \left(\frac{6.450 \times 10^{10}}{4.352 \times 10^{10}}\right) = \ln (1.482) = 0.3934, \quad \text{so} \quad t = \frac{0.3934}{\lambda} = \frac{0.3934}{0.1551 \text{ Gy}^{-1}} = 2.536 \text{ Gy}.
$$

(d) By comparing $^{235}\text{U}$ and $^{207}\text{Pb}$, estimate the age of this rock.

We proceed exactly as before, to obtain

$$
\lambda t = \ln \left(\frac{3.572 \times 10^{9}}{2.940 \times 10^{8}}\right) = \ln (12.64) = 2.497, \quad \text{so} \quad t = \frac{2.497}{\lambda} = \frac{2.497}{0.9846 \text{ Gy}^{-1}} = 2.536 \text{ Gy}.
$$

The fact that the answer came out the same in each case gives us confidence that it is right.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Parent</th>
<th>Daughter</th>
<th>$Q$ (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>e.c.</td>
<td>$^{231}\text{U}$</td>
<td>$^{231}\text{Pa}$</td>
<td>0.358</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$^{231}\text{Pa}$</td>
<td>$^{227}\text{Ac}$</td>
<td>5.195</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$^{227}\text{Ac}$</td>
<td>$^{227}\text{Th}$</td>
<td>0.045</td>
</tr>
</tbody>
</table>
37. Photocopied with the equations on the next page is a portion of Appendix A from the text. $^{231}$U decays via electron capture, then $\alpha$ decay, then $\beta^-$ decay.

(a) For each step, identify the daughter nucleus, and calculate the $Q$-value for the decay. You may place your final answers in the summary at right.

For electron capture, $Z$ decreases by one while $A$ stays the same, so it becomes $^{231}$Pa. The energy released is

$$Q = (m_p - m_N)c^2 = (231.036264 - 231.035880)uc^2 = (0.000385)(931.5 \text{ MeV})$$

$$= 0.358 \text{ MeV}.$$  

For the subsequent $\alpha$ decay, the parent is now the daughter from the previous interaction, or $^{231}$Pa. This time, $Z$ decreases by two while $A$ decreases by four, so it becomes $^{227}$Ac. The energy released is

$$Q = (m_p - m_X - m_{He})c^2 = (231.035884 - 227.027752 - 4.002602)uc^2$$

$$= (0.005577)(931.5 \text{ MeV}) = 5.195 \text{ MeV}.$$  

For the subsequent $\beta^-$ decay, $Z$ increases by one while $A$ stays the same, so it ends at $^{227}$Th. The energy released is

$$Q = (m_p - m_N)c^2 = (227.027749 - 227.027701)uc^2 = (0.000048)(931.5 \text{ MeV})$$

$$= 0.045 \text{ MeV}.$$  

(b) For the first step, the book incorrectly lists the decay as $\beta^+$ decay. Explain why this cannot be correct.

If the first decay were $\beta^+$ decay, $Z$ still decreases by one while $A$ stays the same, so it would still become $^{231}$Pa. However, the formula for $Q$ is slightly different, namely

$$Q = (m_p - m_N)c^2 - 2m_e c^2 = (0.358 \text{ MeV}) - (1.022 \text{ MeV}) = -0.664 \text{ MeV}.$$  

Since the result is negative, this reaction cannot occur.
38. Two stars each have the same mass $M = 0.980 M_\odot = 1.949 \times 10^{30} \text{ kg}$ and have the same surface temperature $T = 5740 \text{ K} = 0.993 T_\odot$.

(a) What is the flux of light from the surface of these stars, in W/m$^2$?

This is a straightforward application of the Stefan-Boltzmann law, which says the flux is given by

$$\mathcal{F} = \sigma T^4 = \left(5.6704 \times 10^{-8} \text{ W m}^{-2} \text{K}^{-4}\right)(5740 \text{ K})^4 = 6.155 \times 10^7 \text{ W/m}^2.$$ 

(b) Star X has a radius of $R_X = 6.80 \times 10^8 \text{ km}$.

Find its luminosity in solar luminosities ($L_\odot = 3.846 \times 10^{26} \text{ W}$).

Flux is luminosity per unit area, so we have $L = A \mathcal{F} = 4\pi R^2 \mathcal{F}$. Substituting in the flux from part (a), we find

$$L_X = A \mathcal{F} = 4\pi R_X^2 \mathcal{F} = 4\pi \left(6.80 \times 10^8 \text{ m}\right)^2 \left(6.155 \times 10^7 \text{ W/m}^2\right) = 3.576 \times 10^{26} \text{ W},$$

$$\frac{L_X}{L_\odot} = \frac{3.576 \times 10^{26} \text{ W}}{3.846 \times 10^{26} \text{ W}} = 0.930, \text{ or } L_X = 0.930 L_\odot.$$ 

(c) Star Y has luminosity $L_Y = 7.53 \times 10^{-5} L_\odot = 2.90 \times 10^{22} \text{ W}$. What is the radius of star X?

We again start with the formula $L = A \mathcal{F} = 4\pi R^2 \mathcal{F}$, which we solve for the radius to give

$$R^2 = \frac{L}{4\pi \mathcal{F}},$$

$$R_X = \sqrt{\frac{L_X}{4\pi \mathcal{F}}} = \sqrt{\frac{2.90 \times 10^{22} \text{ W}}{4\pi (6.155 \times 10^7 \text{ W/m}^2)}} = 6.12 \times 10^8 \text{ m} = 6120 \text{ km}.$$ 

(d) Based on the information you have obtained, and what you know about stars, which type of star are each of these (main sequence, white dwarf, or neutron star)?

A main sequence star with a mass so close to the Sun should have a temperature, radius, and luminosity close to that of the Sun, and indeed, that’s exactly what we have for star X. It is reasonable to conclude that star X is a main sequence star. A white dwarf should have a maximum mass of 1.4 solar masses, and should be about the size of the Earth, which matches very well with star Y. A neutron star should have a mass somewhere in the range of 1.4 to 3 solar masses, and should be about the size of a city. Neither star matches that.

X is a main sequence star
Y is a white dwarf star
39. Two spacecraft are studying a black hole. One is at \( r = 257 \text{ km} \), and one is at an unknown distance less than 25.7 km. The two spacecraft each send signals to Earth with wavelength \( \lambda_0 = 0.147 \text{ m} \). Earth detects two signals, one at a wavelength of \( \lambda_1 = 0.163 \text{ m} \) and the other at \( \lambda_2 = 0.214 \text{ m} \)

(a) Which signal is from the more distant spacecraft, at \( r = 257 \text{ km} \)?

Time slows down, which causes a red shift, or lengthening of the wavelength. The closer you are to the gravity source, the greater the red shift. Hence the longer wavelength is closer in, and \( \lambda_1 = 0.163 \text{ m} \) is from the spacecraft at 257 km.

(b) What is the Schwarzschild radius of the black hole?

We use the formula \( \lambda = \lambda_0 \left( 1 - \frac{2GM}{c^2r} \right)^{-1/2} \), rearrange and solve for \( 2GM/c^2 \):

\[
\sqrt{1 - \frac{2GM}{c^2r}} = \frac{\lambda_0}{\lambda},
\]

\[
\frac{2GM}{c^2r} = 1 - \frac{\lambda^2}{\lambda_0^2} = 1 - \left( \frac{0.147 \text{ m}}{0.163 \text{ m}} \right)^2 = 0.187,
\]

\[
\frac{2GM}{c^2} = 0.187r = 0.187(25.7 \text{ km}) = 48.0 \text{ km} = R_s.
\]

(c) What is the mass of the black hole, in solar masses (\( M_\odot = 1.989 \times 10^{30} \text{ kg} \))?

\[
M = \frac{c^2}{2G} (48.0 \text{ km}) = \left( \frac{2.998 \times 10^8 \text{ m/s}^2}{2(6.673 \times 10^{-11} \text{ m}^3 / \text{kg} / \text{s}^2)} \right) \left( \frac{3.23 \times 10^{30} \text{ kg}}{1.989 \times 10^{30} \text{ kg}} \right) M_\odot = 16.2 M_\odot.
\]

(d) What is the distance of the other spacecraft from the black hole?

We use some of the same equations as before, so we have

\[
\frac{2GM}{c^2r} = 1 - \frac{\lambda_0^2}{\lambda_2^2} = 1 - \left( \frac{0.147 \text{ m}}{0.214 \text{ m}} \right)^2 = 0.528 = \frac{48.0 \text{ km}}{r},
\]

\[
r = \frac{48.0 \text{ km}}{0.528} = 90.8 \text{ km}.
\]
Equations

\[ h = 6.626 \times 10^{-34} \text{ J} \cdot \text{s} = 4.136 \times 10^{-15} \text{ eV} \cdot \text{s} \]
\[ \hbar = 1.055 \times 10^{-34} \text{ J} \cdot \text{s} = 6.582 \times 10^{-16} \text{ eV} \cdot \text{s} \]

u = 931.494 MeV / c^2
u = 1.661 \times 10^{-27} \text{ kg}

Constants:

\[ G = 6.673 \times 10^{-11} \text{ m}^3 / \text{ kg} / \text{s}^2 \]
\[ N_A = 6.022 \times 10^{23} \]
\[ 2m_e c^2 = 1.022 \text{ MeV} \]
\[ M_{\text{He}} = 4.002602 \text{ u} \]
\[ \sigma = 5.6704 \times 10^{-8} \text{ W m}^{-2} \text{K}^{-4} \]

Energy and momentum in relativity:

\[ E = \gamma m c^2 \quad \vec{p} = \gamma m \vec{v} \]

Hydrogen like atoms:

\[ E_n = -\frac{(13.6 \text{ eV}) Z^2}{n^2} \]

Gravitational red shift:

\[ \lambda = \lambda_0 \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} \]

Schwarzschild radius:

\[ R_s = \frac{2GM}{c^2} \]

Expectation values:

\[ \langle \mathcal{O} \rangle = \int_{-\infty}^{\infty} \psi^*(x) \mathcal{O} \psi(x) \, dx \]

Uncertainty:

\[ (\Delta \mathcal{O})^2 = \langle \mathcal{O}^2 \rangle - \langle \mathcal{O} \rangle^2 \]

Isotope Masses