Quantum Mechanics 741 - Final Equations

The following new equations you should memorize, and understand how to use them:

Angular momentum
\[ [J_x, J_y] = i\hbar J_z, \text{ etc.} \]
\[ J^2 \ket{j, m} = \hbar^2 (j^2 + j) \ket{j, m} \]
\[ J_z \ket{j, m} = \hbar m \ket{j, m} \]
\[ j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \]
\[ m = j, j - 1, j - 2, \ldots, -j \]

Examples of Angular Momentum-Like Operators
\[ L, S, J = L + S \]

Spin Commutes
\[ [S_i, R_{j_2}] = 0 \]
\[ [S_i, P_{j_2}] = 0 \]

Addition of Angular Momentum
\[ j = j_1 - j_2, j_1 - j_2 + 1, \ldots, j_1 + j_2 \]

State Operator
\[ \rho = \sum_i f_i \ket{\psi_i} \bra{\psi_i} \]
\[ \text{Tr}(\rho) = 1, \quad \rho^\dagger = \rho \]
Eigenvalues: \( \rho_i \geq 0 \)
\[ \bra{A} = \text{Tr}(\rho A) \]

Examples of Clebsch-Gordan:
\[ \bra{j_1, j_2; m_1, m_2} j, m \rangle \]
non-zero only if:
\[ |j_1 - j_2| \leq j \leq j_1 + j_2 \]
\[ m_1 + m_2 = m, \quad |m| \leq j \]
\[ |m_1| \leq j_1, \quad |m_2| \leq j_2 \]

Adding Angular Momentum
\[ j = j_1 - j_2, j_1 - j_2 + 1, \ldots, j_1 + j_2 \]

Examples of Clebsch-Gordan:
\[ \left\{ \psi_x(r) \right\}_{j, m} = \left\{ \psi_{x-1}(r) \right\}_{j, m} \]

Wave Function with Spin
\[ \psi(r, t) = \left\{ \psi_x(r) \right\}_{j, m} \]

Spin of proton, neutron, electron: \( \frac{1}{2} \)

Kinematic Momentum
\[ \pi = \mathbf{P} + e\mathbf{A} \]

Examples of Clebsch-Gordan:
\[ \left\{ \psi_x(r) \right\}_{j, m} = \left\{ \psi_{x-1}(r) \right\}_{j, m} \]

Vector Operators
\[ \left[ J_x, V_y \right] = i\hbar V_z \]
\[ \left[ J_y, V_x \right] = -i\hbar V_y \]
\[ \left[ J_z, V_x \right] = 0 \text{ etc.} \]

Trace:
\[ \text{Tr}(A) \equiv \sum_i \langle \phi_i | A | \phi_i \rangle \]
\[ \text{Tr}(AB) = \text{Tr}(BA) \]

Examples of Clebsch-Gordan:
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Other things you should know:
- Understanding the concept of spin
- Addition of angular momentum, and when you can use it
- Especially, the resulting angular momentum when you add two angular momenta
- Clebsch-Gordan coefficients:
  - When they don’t vanish
  - How you can use them to explicitly add angular momenta
  - How you can use them in the Wigner-Eckart theorem
  - How you can use them to do integrals of three Spherical Harmonics
- How a vector or scalar commutes with \( J \)
- (in principle) how a spherical tensor commutes with \( J \)
- Gauge transformations do not change the physics, and therefore theories must be gauge invariant
- (qualitatively) why certain gauge choices are better than others
- Be able to describe the Aharonov-Bohm experiment
- Tensor product spaces
- Generically, what a wave function for multiple particles looks like
- How to find exact eigenstates for \( N \) interchangeable non-interacting particles
- How to make these eigenstates completely symmetric/anti-symmetric for bosons/fermions
- The spin-statistics theorem
How the degeneracy pressure calculation is done for fermions
Generally, how different electrons are filled into orbitals for atoms
The time evolution operator is linear and unitary; this allows you to prove things from it
How to use the propagator to get the wave function at time $t$ given it at time $t_0$
How to get the state operator as a matrix, or to write it in terms of basis vectors
How to tell if a state operator is legal; how to tell if it is a pure or mixed state
How to evolve the state operator and use it for evaluating expectation values
In the Heisenberg formalism, state vectors don’t change, but operators do

The following new equations you need not memorize, but you should know how to use them if given to you:

<table>
<thead>
<tr>
<th>Rotating with Spin</th>
<th>Gauge Transformations</th>
<th>Ehrenfest for E&amp;M:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R(R)\psi(r) = D(R)\psi(R^T r)$</td>
<td>$A' = A + \nabla \chi$</td>
<td>$\frac{d}{dt}\langle R \rangle = \frac{1}{m}\langle \pi \rangle$</td>
</tr>
<tr>
<td>$D(R(\hat{n}, \theta)) = \exp(-i \hat{n} \cdot \mathbf{S}/\hbar)$</td>
<td>$U' = U - \partial \chi / \partial t$</td>
<td>$\frac{d}{dt}\langle \pi \rangle = -\frac{e}{2m}\langle \pi \times \mathbf{B} - \mathbf{B} \times \pi \rangle - e\langle \mathbf{E} \rangle$</td>
</tr>
<tr>
<td>$R(R(\hat{n}, \theta)) = \exp(-i \theta \hat{n} \cdot \mathbf{J}/\hbar)$</td>
<td></td>
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<thead>
<tr>
<th>EM Hamiltonian</th>
<th>EM Fields</th>
<th>Fermi Energy and Pressure:</th>
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<tbody>
<tr>
<td>$H = \frac{1}{2m}\pi^2 - eU + \frac{ge^2}{2m}\mathbf{B} \cdot \mathbf{S}$</td>
<td>$\mathbf{B} = \nabla \times \mathbf{A}$</td>
<td>$E_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$</td>
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<tr>
<td>$\mathbf{E} = -\partial \mathbf{A} / \partial t - \nabla U$</td>
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<tr>
<th>Wigner-Eckart Theorem</th>
<th>Integral of Spherical Harmonics: Non-zero requires $l_1 + l_2 - l$ even</th>
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<tr>
<td>$\langle \alpha, j, m \mid T_q^{(k)} \mid \alpha', j', m' \rangle = \frac{1}{\sqrt{2j + 1}} \langle \alpha, j \mid T_q^{(k)} \mid \alpha', j' \rangle \langle j', k; m', q \mid j, m \rangle$</td>
<td>$\int d\Omega \ Y_{l_1}^{m_1}(\theta, \phi)^* Y_{l_2}^{m_2}(\theta, \phi) Y_{l_3}^{m_3}(\theta, \phi) = \sqrt{\frac{(2l_1 + 1)(2l_2 + 1)}{4\pi}} \langle l_1, l_2, 0, 0 \mid l_1, l_2, m_1, m_2 \rangle \langle l_1, l_2, m_1, m_2 \mid l_1, l_2, 0, 0 \rangle$</td>
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<th>Propagator:</th>
<th>Free Propagator (1D):</th>
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<td>$\Psi(r, t) = \int d^3 r_0 K(r, t; r_0, t_0) \Psi(r_0, t_0)$</td>
<td>$K(x, t; x_0, t_0) = \frac{m}{\sqrt{2\pi i\hbar(t-t_0)}} \exp \left[ \frac{im(x-x_0)^2}{2\hbar(t-t_0)} \right]$</td>
</tr>
<tr>
<td>$K(r, t_0, t_0) = \sum_n \phi_n^*(r) e^{-iE_n(t-t_0)/\hbar} \phi_n(r_0)$</td>
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<th>State Operators:</th>
<th>Heisenberg Picture</th>
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<td>$ih \frac{d}{dt} \rho = [H, \rho]$</td>
<td>$\frac{d}{dt} A(t) = \frac{i}{\hbar} [H(t), A(t)]$</td>
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