Quantum Mechanics
Solutions to Graduate Exam

Each problem is worth 25 points. The points for individual parts are marked in square brackets. Some useful formulas are given at the end of the exam. To ensure full credit, show your work. Do any four (4) of the following five (5) problems. If you attempt all 5 problems you must clearly state which 4 problems you want to have graded.

1. A two-state quantum system has Hamiltonian given by

\[ H = \hbar \omega \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}, \]

where \( \omega \) is a constant.

(a) [8 points] Find the eigenstates and normalized eigenvectors of this Hamiltonian.

We start by pulling out the common factor of \( \hbar \omega \), and then try to find the eigenvalues of the remaining matrix (which must be multiplied by \( \hbar \omega \)) and corresponding eigenvectors (which do not). We subtract \( \lambda \) times the identity matrix and set the determinant to zero, to yield

\[ 0 = \det \begin{pmatrix} 3 - \lambda & 4 \\ 4 & -3 - \lambda \end{pmatrix} = (3 - \lambda)(-3 - \lambda) - 4 \cdot 4 = -9 + 3\lambda - 3\lambda + \lambda^2 - 16 = \lambda^2 - 25, \]

\[ \lambda = \pm 5. \]

The eigenvectors can then be found by solving the equation

\[ \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm 5 \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \]

\[ 3\alpha + 4\beta = \pm 5\alpha \quad \text{and} \quad 4\alpha - 3\beta = \pm 5\beta, \]

\[ (3 \mp 5)\alpha = -4\beta \quad \text{and} \quad 4\alpha = (3 \pm 5)\beta. \]

We now need to work out each of the cases. If we take the upper sign, the equations become

\[ -2\alpha = -4\beta \quad \text{and} \quad 4\alpha = 8\beta, \]

which are identical equations, so \( \alpha = 2\beta \). Getting the normalization right demands that \( 1 = \alpha^2 + \beta^2 = 5\beta^2 \), which then gives you your first normalized eigenstate. If we take the lower sign, then equations become \( 8\alpha = -4\beta \quad \text{and} \quad 4\alpha = -2\beta, \) so \( \beta = -2\alpha \).

Normalizing again demands \( 1 = \alpha^2 + \beta^2 = 5\alpha^2 \). So we have our two eigenvalues and eigenvectors:

\[ |+\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad |-\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \]

These states form a complete orthonormal set. The eigenvalues are \( \pm 5\hbar \omega \).
(b) [8 points] The system starts in the state $|\Psi(t = 0)\rangle = |1\rangle = \left(\begin{array}{c} 1 \\ 0 \end{array}\right)$. Find $|\Psi(t)\rangle$.

To do so, we first have to write $|\Psi(t = 0)\rangle$ in terms of our eigenstates. This is pretty straightforward, since we can use the fact that $|\pm\rangle$ is an orthonormal set:

$$|\Psi(t = 0)\rangle = |+\rangle \langle +| \Psi(t = 0)\rangle + |-\rangle \langle -| \Psi(t = 0)\rangle = \frac{1}{\sqrt{5}} (2|+\rangle + |-\rangle).$$

Each eigenstate can then be evolved as a function of time, so that it is multiplied by $e^{-iEt/\hbar}$, so we have

$$|\Psi(t)\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-5i\omega t} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{5i\omega t} = \frac{1}{5} \begin{pmatrix} 4e^{-5i\omega t} + e^{5i\omega t} \\ 2e^{-5i\omega t} - 2e^{5i\omega t} \end{pmatrix}.$$

(c) [5 points] At time $t$ what is the probability that it is in state $|1\rangle$?

The probability is simply

$$|\langle 1|\Psi(t)\rangle|^2 = \frac{1}{25} \left| 4e^{-5i\omega t} + e^{5i\omega t} \right|^2 = \frac{1}{25} \left( 4e^{-5i\omega t} + e^{5i\omega t} \right) \left( 4e^{5i\omega t} + e^{-5i\omega t} \right)$$

$$= \frac{1}{25} \left( 16 + 4e^{-10i\omega t} + 4e^{10i\omega t} + 25 \right) = \frac{17}{25} + \frac{8}{25} \cos(10\omega t).$$

(d) [4 points] At time $t$, the energy is measured and found to be positive. What is the state vector after this measurement?

Because the energy is positive, it must be in the positive energy state, or $|+\rangle$.

Technically, we should extract that portion of the state vector that has the correct energy and then renormalize, so that

$$|\Psi_{\text{after}}\rangle = \frac{1}{\sqrt{P(+)}} |+\rangle \langle +| \Psi_{\text{before}}\rangle = \sqrt{\frac{5}{4} |+\rangle} \frac{2}{\sqrt{5}} e^{-5i\omega t} = |+\rangle e^{-5i\omega t} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5i\omega t}.$$

The phase is, however, irrelevant.
2. A hydrogen atom has its electron in a superposition of various states:

\[ |\psi\rangle = N \left( -i \left| 1, 0, 0, -\frac{1}{2} \rightangle + \sqrt{2} \left| 2, 0, 0, -\frac{1}{2} \rightangle + \left| 2, 1, +1, +\frac{1}{2} \rightangle \right), \]

in the notation where the quantum numbers are \( |n, l, m_l, m_s\rangle \) where \( n \) is the principal quantum number, \( l \) is the quantum number associated with the operator \( L^2 \), \( m_l \) is the quantum number associated with \( L_z \), and \( m_s \) is the quantum number associated with \( S_z \).

(a) [2 points] What is the normalization \( N \)?

\[
1 = \langle \psi | \psi \rangle = |N|^2 \left( i \left| 1, 0, 0, -\frac{1}{2} \rightangle + \sqrt{2} \left| 2, 0, 0, -\frac{1}{2} \rightangle + \left| 2, 1, +1, +\frac{1}{2} \rightangle \right) \cdot \left( -i \left| 1, 0, 0, -\frac{1}{2} \rightangle + \sqrt{2} \left| 2, 0, 0, -\frac{1}{2} \rightangle + \left| 2, 1, +1, +\frac{1}{2} \rightangle \right) \\
= |N|^2 \left( 1 + 2 + 1 \right) = 4 |N|^2 ,
\]

\[ |N| = \frac{1}{2} . \]

The phase is, of course, unknown and irrelevant.

(b) [8 points] For each of the following operators, if we measured any one of them, what would be the possible outcomes and probabilities: \( H, L_z, S_z, L^2, S^2 \)?

All of these states are manifestly eigenstates of these. For the Hamiltonian, the energy is given by the formula for \( E_n \), given at the end of the test. For \( L_z \) and \( S_z \), the eigenvalues are \( \hbar m_l \) and \( \hbar m_s \) respectively. For \( L^2 \) and \( S^2 \), the eigenvalues are \( \hbar (l^2 + \ell) \) and \( \hbar (s^2 + s) \), but \( s = \frac{1}{2} \) for the electron, so there is never any uncertainty in this quantity. For all the others, the probabilities are just the (sum of the) amplitudes of the relevant parts (after substituting \( N = \frac{1}{2} \)), so we have

\[
H : \quad P(-13.6 \text{ eV}) = \frac{1}{4} , \quad P(-3.4 \text{ eV}) = \frac{3}{4} ,
\]

\[
L_z : \quad P(0) = \frac{3}{4} , \quad P(\pm \hbar) = \frac{1}{4} ,
\]

\[
S_z : \quad P\left( \pm \frac{1}{2} \hbar \right) = \frac{1}{4} , \quad P\left( \pm \frac{1}{2} \hbar \right) = \frac{3}{4} ,
\]

\[
L^2 : \quad P(0) = \frac{3}{4} , \quad P(2\hbar^2) = \frac{1}{4} ,
\]

\[
S^2 : \quad P\left( \frac{3}{4} \hbar^2 \right) = 1 .
\]
(c) [8 points] Define the total angular momentum \( J = L + S \). If we measure \( J_z \), what would be the possible outcomes and corresponding probabilities? If we measure \( J^2 \), what would be the possible outcomes and corresponding probabilities?

It is obvious that each of these states is an eigenstate of \( J_z \). The first two have eigenvalue \(-\frac{1}{2} \hbar\), while the last one has eigenvalue \(+\frac{3}{2} \hbar\).

Trickier is realizing that these states are also eigenvalues of \( J^2 \). For the first two terms, you are adding \( l = 0 \) to \( s = \frac{1}{2} \), which can only result in \( j = \frac{1}{2} \). For the last term, you are adding \( l = 1 \) to \( s = \frac{1}{2} \), which generally can lead to \( j = \frac{1}{2} \) or \( j = \frac{3}{2} \). But since \( m_j = \frac{3}{2} \), and we must have \( |m_j| \leq j \), then we must have \( j = \frac{3}{2} \). Hence they are all eigenstates of \( J^2 \) as well, with eigenvalues \( \hbar^2 \left(j^2 + j\right) \). So we have

\[
J_z: \quad P\left(-\frac{1}{2} \hbar\right) = \frac{3}{4}, \quad P\left(+\frac{3}{2} \hbar\right) = \frac{1}{4},
\]

\[
J^2: \quad P\left(\frac{3}{4} \hbar^2\right) = \frac{3}{4}, \quad P\left(\frac{15}{4} \hbar^2\right) = \frac{1}{4}.
\]

(d) [7 points] Simplify, as much as possible, the expectation value of the three position operators, \( \langle \Psi | R | \Psi \rangle \), for this state.

To work this out, we must, in principle, write out

\[
\langle \Psi | R | \Psi \rangle = \frac{1}{4} \left( i \langle 1, 0, 0, -\frac{1}{2} | + \sqrt{2} \langle 2, 0, 0, -\frac{1}{2} | + \langle 2, 1, +1, +\frac{1}{2} | \right) R
\]

\[
\cdot \left( -i \langle 1, 0, 0, -\frac{1}{2} | + \sqrt{2} \langle 2, 0, 0, -\frac{1}{2} | \right) R
\]

Now, what terms might contribute? Because the position operators commute with spin, they can’t change the spin, and hence only those terms that leave the spin intact might contribute:

\[
\langle \Psi | R | \Psi \rangle = \frac{1}{4} \left( i \langle 1, 0, 0, -\frac{1}{2} | + \sqrt{2} \langle 2, 0, 0, -\frac{1}{2} | \right) R \left( -i \langle 1, 0, 0, -\frac{1}{2} | + \sqrt{2} \langle 2, 0, 0, -\frac{1}{2} | \right)
\]

\[
+ \frac{1}{4} \langle 2, 1, +1, +\frac{1}{2} | R \langle 2, 1, +1, +\frac{1}{2} \rangle
\]

However, it is also known that dipole moments like these are non-zero only if \( l \) changes by one. Hence all the terms on the right vanish, and we conclude \( \langle \Psi | R | \Psi \rangle = 0 \).
3. Lame Larry is attempting to find the energy of the ground state of the 3D harmonic oscillator with Hamiltonian $H = \frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 r^2$. Unaware that it has an exact solution, he attempts to estimate the ground state energy using the variational principle with trial wave function $\psi(r) = \exp\left(-\frac{1}{2} \alpha r\right)$. Find the resulting estimate of the ground state energy. Compare your answer to the true ground state energy.

To calculate this, we have to put three pieces together, which we now calculate:

$$\langle \psi | H | \psi \rangle = \int d^3 r \langle e^{-\alpha r}/2 \rangle^2 = \int d\Omega r^2 dr e^{-\alpha r} = 4\pi \cdot \frac{2}{\alpha^3} = \frac{8\pi}{\alpha^3},$$

$$\langle \psi | P^2 | \psi \rangle = -\hbar^2 \int d^3 r \langle e^{-\alpha r}/2 \rangle \nabla^2 \langle e^{-\alpha r}/2 \rangle = -\hbar^2 \int d\Omega r^2 dr e^{-\alpha r/2} \left[ \frac{\alpha^2}{4} - \frac{\alpha}{r} \right] e^{-\alpha r/2}$$

$$= -4\pi \hbar^2 \int_0^\infty dr \left( \frac{\alpha^2}{4} r^2 - \alpha r \right) e^{-\alpha r} = -4\pi \hbar^2 \left( \frac{2\alpha^2}{4\alpha^3} - \frac{\alpha}{\alpha^2} \right) = \frac{2\pi \hbar^2}{\alpha},$$

$$\langle \psi | R^2 | \psi \rangle = \int d^3 r r^2 \langle e^{-\alpha r}/2 \rangle^2 = \int d\Omega r^4 dr e^{-\alpha r} = 4\pi \frac{24}{\alpha^5} = \frac{96\pi}{\alpha^5}. $$

We then assemble them to get the energy as a function of $\alpha$:

$$E(\alpha) = \frac{\langle \psi | H | \psi \rangle}{\langle \psi | P^2 | \psi \rangle + \frac{1}{2} m \omega^2 \langle \psi | R^2 | \psi \rangle} = \frac{\alpha^3}{8\pi} \left[ \frac{2\pi \hbar^2}{2m\alpha} + \frac{m \omega^2 96\pi}{\alpha^5} \right]$$

$$= \frac{\alpha^2 \hbar^2}{8m} + \frac{6m \omega^2}{\alpha^2}. $$

We now minimize this function by first taking the derivative with respect to $\alpha$:

$$0 = \frac{d}{d\alpha} E(\alpha) = \frac{\alpha \hbar^2}{4m} - \frac{12m \omega^2}{\alpha^3},$$

$$\alpha^4 = \frac{48m^2 \omega^2}{\hbar^2},$$

$$\alpha^2 = \frac{4\sqrt{3}m \omega}{\hbar}. $$

We then substitute it back into the expression to get an estimate of the ground state:

$$E_0 \approx \frac{\hbar^2}{8m} \left( \frac{4\sqrt{3}m \omega}{\hbar} \right) + 6m \omega^2 \left( \frac{\hbar}{4\sqrt{3}m \omega} \right) = \hbar \omega \left( \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \right) = \sqrt{3} \hbar \omega \approx 1.732 \hbar \omega. $$

The exact answer is $E_0 = 1.5 \hbar \omega$, which is slightly less than the approximation, as it must be.
4. Consider the two-dimensional harmonic oscillator, both dimensions having classical angular frequency $\omega$, and neglect spin.

(a) [4 points] For a single particle, what are the energy eigenvalues of this Hamiltonian?

The resulting Hamiltonian is just $H = H_x + H_y$, and eigenstates of these Hamiltonians can be found separately. Therefore the states can be labeled $|n_x, n_y\rangle$, and the energies will just be sums, so

$$E_{n_x,n_y} = \hbar \omega (n_x + \frac{1}{2}) + \hbar \omega (n_y + \frac{1}{2}) = \hbar \omega (n_x + n_y + 1).$$

(b) [6 points] Consider two non-interacting distinguishable particles in this oscillator. Write down the general two-particle wave function and corresponding energies in terms of the single particle wave functions $\psi_{n_x,n_y}(r_1)$ and $\psi_{n_x,n_y}(r_2)$.

In general, the wave functions will just be products, and the energies sums, so we have

$$\psi_{n_x,n_y,n_x',n_y'}(r_1,r_2) = \psi_{n_x,n_y}(r_1)\psi_{n_x',n_y'}(r_2).$$

The corresponding energy will be

$$E_{n_x,n_y,n_x',n_y'} = E_{n_x,n_y} + E_{n_x',n_y'} = \hbar \omega (n_x + n_y + n_x' + n_y' + 2).$$

(c) [6 points] Same as (b), but assume identical bosons and write down all degenerate wave functions for the first excited state. What is the degeneracy and the corresponding energy?

The wave functions are the same, but they must be symmetrized. Now, if $(n_{x1},n_{y1}) = (n_{x2},n_{y2})$, then there is no need to symmetrize, but otherwise we have

$$\psi_{n_x,n_y,n_x',n_y'}(r_1,r_2) = \frac{1}{\sqrt{2}}\left[\psi_{n_x,n_y}(r_1)\psi_{n_x',n_y'}(r_2) + \psi_{n_x,n_y}(r_2)\psi_{n_x',n_y'}(r_1)\right].$$

The ground state, $\psi_{0000}$, is already symmetrized, but the first excited state is not. So this state is

$$\psi_{0001}(r_1,r_2) = \frac{1}{\sqrt{2}}\left[\psi_{00}(r_1)\psi_{01}(r_2) + \psi_{01}(r_1)\psi_{00}(r_2)\right],$$

$$\psi_{0010}(r_1,r_2) = \frac{1}{\sqrt{2}}\left[\psi_{00}(r_1)\psi_{10}(r_2) + \psi_{10}(r_1)\psi_{00}(r_2)\right].$$

These states are degenerate and have energy $3\hbar \omega$. There is a degeneracy of two states.

(d) [9 points] Same as (b), but assume identical fermions and write down all degenerate wave functions for the ground state. What is the degeneracy and the corresponding energy? What is the degeneracy and energy of the first excited state?

This time we need to anti-symmetrize the two states, so we have

$$\psi_{n_x,n_y,n_x',n_y'}(r_1,r_2) = \frac{1}{\sqrt{2}}\left[\psi_{n_x,n_y}(r_1)\psi_{n_x',n_y'}(r_2) - \psi_{n_x,n_y}(r_2)\psi_{n_x',n_y'}(r_1)\right].$$
However, if \( (n_{x1}, n_{y1}) = (n_{x2}, n_{y2}) \), then the state is not already anti-symmetrized, and indeed, attempting to do so yields zero. Hence states like \( \psi_{0000} \) simply do not exist. Hence our ground state is degenerate, and given by

\[
\psi_{0001}(r_1, r_2) = \frac{1}{\sqrt{2}} \left[ \psi_{00}(r_1) \psi_{01}(r_2) - \psi_{01}(r_1) \psi_{00}(r_2) \right], \\
\psi_{0010}(r_1, r_2) = \frac{1}{\sqrt{2}} \left[ \psi_{00}(r_1) \psi_{10}(r_2) - \psi_{10}(r_1) \psi_{00}(r_2) \right].
\]

Of course, these still have energy \( 3\hbar \omega \).

For the first excited state, we can’t go to \( \psi_{0101} \) or \( \psi_{1010} \), but we can go to \( \psi_{0002} \), \( \psi_{0011} \), \( \psi_{0120} \) or \( \psi_{0110} \). These states have energy \( 4\hbar \omega \), and, of course, a degeneracy of four.

5. A particle of mass \( m \) is confined in a one-dimensional infinite square well, with allowed region \( 0 < x < L \).
   (a) [4 points] What are the energy eigenvalues and properly normalized eigenstate wave functions of the particle?

The states must be eigenstates of the Hamiltonian, which in the allowed region is simply

\( H = p^2/2m \), so we have

\( E\psi = H\psi = \frac{1}{2m} p^2 \psi = -\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} \)

Hence the function must be proportional to its own second derivative, which implies a function of the form \( \psi \sim \sin(kx) \) or \( \psi \sim \cos(kx) \). But because the potential is infinite at \( x = 0 \) and \( x = L \), the wave function must vanish, so we must use \( \sin(kx) \), and we must have \( \sin(kL) = 0 \), so \( k = \pi n/L \). This implies the wave function must be of the form \( \psi_n(x) = N \sin(\pi nx/L) \). The normalization is fixed by demanding that

\[
1 = \int_0^L N^2 \sin^2\left(\frac{\pi nx}{L}\right) dx = N^2 \left[ \frac{x}{2} + \frac{L}{4\pi n} \cos\left(\frac{2\pi nx}{L}\right) \right]_0^L = N^2 \left[ \frac{L}{2} + \frac{L}{4\pi n} (1-1) \right] = LN^2/2.
\]

It is then an easy matter to determine the energy by direct substitution. So our normalized wave functions and energy are

\[
\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi nx}{L}\right), \quad E_n = \frac{\pi^2 \hbar^2 n^2}{2mL^2}.
\]

(b) [6 points] The particle is in its lowest possible energy state when the separation between the walls is slowly (adiabatically) increased to \( 2L \). How does the expectation value of the energy change?

The process is described as the adiabatic approximation. The quantum state therefore, since it started in the lowest energy state, will still be in the lowest energy state, and hence is in
the ground state. The initial energy was $E_1 = \frac{\pi^2 h^2}{2mL^2}$, but because $L$ has doubled, the energy drops by a factor of 4. So the change in energy is

$$\Delta E = E'_1 - E_1 = \frac{\pi^2 h^2}{2m(2L)^2} - \frac{\pi^2 h^2}{2mL^2} = -\frac{3\pi^2 h^2}{8mL^2}.$$  

(c) [6 points] Again, let the particle start out in its lowest energy state, trapped between the walls separated by $L$. Now let the right-hand wall move instantaneously to the right, increasing the distance to $2L$. How much has the expectation value of the energy changed immediately afterwards?

In the sudden approximation, the Hamiltonian changes so quickly that the quantum state cannot change. But the Hamiltonian in the allowed region is still $H = P^2 / 2m$, and hence it will still have the same expectation value of the energy. Hence the change in energy is $\Delta E = 0$.

(d) [9 points] As in part (c), when the wall is moved to the right, what is the probability that it will end up in the ground state?

The probability of it remaining in the ground state is the square of the overlap of the ground state vectors before and after, so that

$$P(1 \rightarrow 1') = |\langle \psi_1' | \psi_1 \rangle|^2 = \left| \int_0^L \sqrt{\frac{2}{L}} \sqrt{\frac{2}{2L}} \sin \left( \frac{\pi x}{L} \right) \sin \left( \frac{\pi x}{2L} \right) dx \right|^2$$

$$= \frac{2}{L^2} \left[ \frac{1}{2} \frac{2L}{\pi} \sin \left( \frac{\pi x}{2L} \right) - \frac{1}{2} \frac{2L}{3\pi} \sin \left( \frac{3\pi x}{2L} \right) \right]^L_0$$

$$= \frac{2}{L^2} \left[ \frac{L}{\pi} \sin \left( \frac{\pi}{2} \right) - \frac{L}{3\pi} \sin \left( \frac{3\pi}{2} \right) \right]^2 = \frac{2}{L^2} \left( \frac{L}{\pi} + \frac{L}{3\pi} \right)^2 = \frac{2}{L^2} \left( \frac{4L}{3\pi} \right)^2 = \frac{32}{9\pi^2}.$$
Possibly Useful Formulas:

Hydrogen atom: \( E = -\frac{13.6 \text{ eV}}{n^2} \).

Derivative in spherical coordinates:

\[
\nabla \psi = \hat{r} \frac{\partial \psi}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial \psi}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \phi},
\]

\[
\nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2}.
\]

Integrals: For all integrals below, \( n \) is assumed to be a positive integer, and \( \beta \) and \( \alpha \) are assumed to be positive, and \( \alpha \neq \beta \).

\[
\int_0^{\infty} x^n e^{-\beta x} \, dx = \frac{n!}{\beta^{n+1}},
\]

\[
\int \cos^2 (\alpha x) \, dx = \frac{x}{2} + \frac{1}{4\alpha} \sin (2\alpha x),
\]

\[
\int \sin^2 (\alpha x) \, dx = \frac{x}{2} - \frac{1}{4\alpha} \sin (2\alpha x),
\]

\[
\int \cos (\alpha x) \cos (\beta x) \, dx = \frac{\sin \left[ (\alpha - \beta) x \right]}{2(\alpha - \beta)} + \frac{\sin \left[ (\alpha + \beta) x \right]}{2(\alpha + \beta)},
\]

\[
\int \sin (\alpha x) \sin (\beta x) \, dx = \frac{\sin \left[ (\alpha - \beta) x \right]}{2(\alpha - \beta)} - \frac{\sin \left[ (\alpha + \beta) x \right]}{2(\alpha + \beta)},
\]

\[
\int \sin (\alpha x) \cos (\beta x) \, dx = -\frac{\cos \left[ (\alpha - \beta) x \right]}{2(\alpha - \beta)} - \frac{\cos \left[ (\alpha + \beta) x \right]}{2(\alpha + \beta)}.
\]