Each problem is worth 25 points. The points for individual parts are marked in square brackets. Some useful formulas are given at the end of the exam. To ensure full credit, show your work. Do any four (4) of the following five (5) problems. If you attempt all 5 problems you must clearly state which 4 problems you want to have graded.

1. A two-state quantum system has Hamiltonian given by

\[ H = \hbar \omega \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}, \]

where \( \omega \) is a constant.

(a) [8 points] Find the eigenstates and normalized eigenvectors of this Hamiltonian.

(b) [8 points] The system starts in the state \( |\Psi(t = 0)\rangle = |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \). Find \( |\Psi(t)\rangle \).

(c) [5 points] At time \( t \) what is the probability that it is in state \( |1\rangle \)?

(d) [4 points] At time \( t \), the energy is measured and found to be positive. What is the state vector after this measurement?

2. A hydrogen atom has its electron in a superposition of various states:

\[ |\Psi\rangle = N \left( -i |1,0,0,-\frac{1}{2}\rangle + \sqrt{2} |2,0,0,-\frac{1}{2}\rangle + |2,1,1,\frac{1}{2}\rangle \right), \]

in the notation where the quantum numbers are \( |n,l,m_l,m_s\rangle \) where \( n \) is the principal quantum number, \( l \) is the quantum number associated with the operator \( L^2 \), \( m_l \) is the quantum number associated with \( L_z \), and \( m_s \) is the quantum number associated with \( S_z \).

(a) [2 points] What is the normalization \( N \)?

(b) [8 points] For each of the following operators, if we measured any one of them, what would be the possible outcomes and probabilities: \( H \), \( L_z \), \( S_z \), \( L^2 \), \( S^2 \)?

(c) [8 points] Define the total angular momentum \( J = L + S \). If we measure \( J_z \), what would be the possible outcomes and corresponding probabilities? If we measure \( J^2 \), what would be the possible outcomes and corresponding probabilities?

(d) [7 points] Simplify, as much as possible, the expectation value of the three position operators, \( \langle \Psi | \mathbf{R} | \Psi \rangle \), for this state, without doing any explicit spatial integrals.

3. Lame Larry is attempting to find the energy of the ground state of the 3D harmonic oscillator with Hamiltonian \( H = \frac{1}{2m} \mathbf{P}^2 + \frac{1}{2} \omega \mathbf{R}^2 \). Unaware that it has an exact solution, he attempts to estimate the ground state energy using the variational principle with trial wave function \( \psi(r) = \exp \left( -\frac{1}{2} \alpha r \right) \). Find the resulting estimate of the ground state energy. Compare your answer to the true ground state energy.
4. Consider the two-dimensional harmonic oscillator, both dimensions having classical angular frequency $\omega$, and neglect spin.
(a) [4 points] For a single particle, what are the energy eigenvalues of this Hamiltonian?
(b) [6 points] Consider two non-interacting distinguishable particles in this oscillator. Write down the general two-particle wave function and corresponding energies in terms of the single particle wave functions $\psi_{n_1, n_2}(r_1)$ and $\psi_{n_1, n_2}(r_2)$.
(c) [6 points] Same as (b), but assume identical bosons and write down all degenerate wave functions for the first excited state. What is the degeneracy and the corresponding energy?
(d) [9 points] Same as (b), but assume identical fermions and write down all degenerate wave functions for the ground state. What is the degeneracy and the corresponding energy? What is the degeneracy and energy of the first excited state?

5. A particle of mass $m$ is confined in a one-dimensional infinite square well, with allowed region $0 < x < L$.
(a) [4 points] What are the energy eigenvalues and properly normalized eigenstate wave functions of the particle?
(b) [6 points] The particle is in its lowest possible energy state when the separation between the walls is slowly (adiabatically) increased to $2L$. How does the expectation value of the energy change?
(c) [6 points] Again, let the particle start out in its lowest energy state, trapped between the walls separated by $L$. Now let the right-hand wall move instantaneously to the right, increasing the distance to $2L$. How much has the expectation value of the energy changed immediately afterwards?
(d) [9 points] As in part (c), when the wall is moved to the right, what is the probability that it will end up in the ground state?
Possibly Useful Formulas:

Hydrogen atom: \( E = -\frac{13.6 \text{ eV}}{n^2} \).

Derivative in spherical coordinates:
\[
\nabla \psi = \hat{r} \frac{\partial \psi}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial \psi}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \phi},
\]
\[
\nabla^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2}.
\]

Integrals: For all integrals below, \( n \) is assumed to be a positive integer, and \( \beta \) and \( \alpha \) are assumed to be positive, and \( \alpha \neq \beta \).
\[
\int_0^\infty x^n e^{-\beta x} \, dx = \frac{n!}{\beta^{n+1}},
\]
\[
\int \cos^2 (\alpha x) \, dx = \frac{x}{2} + \frac{1}{4\alpha} \sin (2\alpha x),
\]
\[
\int \sin^2 (\alpha x) \, dx = \frac{x}{2} - \frac{1}{4\alpha} \sin (2\alpha x),
\]
\[
\int \cos (\alpha x) \cos (\beta x) \, dx = \frac{\sin \left[ (\alpha - \beta) x \right]}{2(\alpha - \beta)} + \frac{\sin \left[ (\alpha + \beta) x \right]}{2(\alpha + \beta)},
\]
\[
\int \sin (\alpha x) \sin (\beta x) \, dx = \frac{\sin \left[ (\alpha - \beta) x \right]}{2(\alpha - \beta)} - \frac{\sin \left[ (\alpha + \beta) x \right]}{2(\alpha + \beta)},
\]
\[
\int \sin (\alpha x) \cos (\beta x) \, dx = -\frac{\cos \left[ (\alpha - \beta) x \right]}{2(\alpha - \beta)} - \frac{\cos \left[ (\alpha + \beta) x \right]}{2(\alpha + \beta)}.
\]