1. [15] Two spinless particles are in the 1D infinite square well with allowed region $0 < x < a$ are in one of the two states $|\psi\rangle = N (|\phi_1, \phi_2\rangle \pm |\phi_2, \phi_1\rangle)$, where $N$ is a constant and $|\phi_n\rangle$ is the $n$'th eigenstate of the 1D infinite square well.

(a) Determine the normalization constant $N$ and write explicitly the wave function $\psi(x_1, x_2)$.

To get the normalization right, we simply demand that

$$1 = \langle \psi | \psi \rangle = N^2 \left( \langle \phi_1, \phi_2 | \pm \langle \phi_2, \phi_1 | \right) \left( \langle \phi_1, \phi_2 \rangle \pm \langle \phi_2, \phi_1 \rangle \right) = N^2 (1 \pm 0 \pm 0 + 1) = 2N^2,$$

from which we conclude that $N = \frac{1}{\sqrt{2}}$. The wave function, therefore, is

$$\psi(x_1, x_2) = \langle x_1, x_2 | \psi \rangle = \frac{1}{\sqrt{2}} \left( \langle x_1, x_2 | \phi_1, \phi_2 \rangle \pm \langle x_1, x_2 | \phi_2, \phi_1 \rangle \right) = \frac{\phi_1(x_1) \phi_2(x_2) \pm \phi_2(x_1) \phi_1(x_2)}{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{a} \left[ \sin \left( \frac{\pi x_1}{a} \right) \sin \left( \frac{2\pi x_2}{a} \right) \pm \sin \left( \frac{2\pi x_1}{a} \right) \sin \left( \frac{\pi x_2}{a} \right) \right].$$

(b) The positions of the two particles are measured. What is the probability that (i) $0 < x_1 < \frac{1}{2}a$, (ii) $0 < x_2 < \frac{1}{2}a$ and (iii) both are true.

We integrate the square of the wave function over the indicated range, with any unspecified integrals over the entire range from 0 to $a$. The range will depend on which part we are doing. The resulting integrals are given as Eqs. (A.12n) and (A.12q) in the appendix. We will do the indefinite integrals first, and then just substitute the appropriate limits.

$$\int dx_1 \int dx_2 |\psi(x_1, x_2)|^2 = \frac{2}{a^2} \int dx_1 \int dx_2 \left[ \sin \left( \frac{\pi x_1}{a} \right) \sin \left( \frac{2\pi x_2}{a} \right) \pm \sin \left( \frac{2\pi x_1}{a} \right) \sin \left( \frac{\pi x_2}{a} \right) \right]^2$$

$$= \frac{2}{a^2} \int dx_1 \int dx_2 \left[ \sin^2 \left( \frac{\pi x_1}{a} \right) \sin^2 \left( \frac{2\pi x_2}{a} \right) \pm \sin^2 \left( \frac{2\pi x_1}{a} \right) \sin^2 \left( \frac{\pi x_2}{a} \right) \right]$$

$$\pm 2 \left[ \frac{1}{2} \sin \left( \frac{\pi x_1}{a} \right) - \frac{1}{4} \sin \left( 2\pi x_1/a \right) \right] \left[ \frac{1}{2} \sin \left( \frac{\pi x_2}{a} \right) - \frac{1}{4} \sin \left( 2\pi x_2/a \right) \right]$$

$$\pm 4 \left[ \frac{1}{\pi} \sin \left( \frac{\pi x_1}{a} \right) - \frac{1}{4\pi} \sin \left( 3\pi x_1/a \right) \right] \left[ \frac{1}{\pi} \sin \left( \frac{\pi x_2}{a} \right) - \frac{1}{4\pi} \sin \left( 3\pi x_2/a \right) \right]$$

Substituting in our limits, we note that every term vanishes when $x_1 = 0$ or $x_2 = 0$, so we need only include the upper limits. We therefore have:

$$P(x_1 < \frac{1}{2}a) = \int_0^{\frac{1}{2}a} dx_1 \int_0^{\frac{1}{2}a} dx_2 |\psi(x_1, x_2)|^2 = 2 \cdot \frac{1}{4} \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} \cdot \frac{1}{2} \pm 4 \left[ \frac{1}{2\pi} \cdot 1 - \frac{1}{6\pi} \cdot (-1) \right] = \frac{1}{2} = 50\%.$$
2. [10] A system of more than two particles is an eigenstate of every pair switching operator, that is, $P(i \leftrightarrow j)\ket{\psi} = \lambda_{ij} \ket{\psi}$ for every $i \neq j$.

(a) [3] For $i$, $j$, and $k$ all different, simplify the product $P(i \leftrightarrow j) P(i \leftrightarrow k) P(i \leftrightarrow j)$.

If we start on the right, we see that $i$ goes to $j$, then it is left alone, then it goes to $i$ again. We see that $j$ goes to $i$, and then to $k$, and there it stays. Finally, $k$ doesn’t go anywhere at first, then it goes to $i$, and from there to $j$. Putting it all together, we see that

$$P(i \leftrightarrow j) P(i \leftrightarrow k) P(i \leftrightarrow j) = P(j \leftrightarrow k)$$

(b) [4] Demonstrate that $\lambda_{ik} = \lambda_{jk}$ for any $i$, $j$, and $k$ all different.

We assume that our wave function is an eigenstate of all pair switching operators. We have

$$P(i \leftrightarrow j) P(i \leftrightarrow k) P(i \leftrightarrow j) \ket{\psi} = P(j \leftrightarrow k) \ket{\psi}$$

$$\lambda_{ij} \lambda_{ik} \lambda_{ij} \ket{\psi} = \lambda_{jk} \ket{\psi}$$

$$\lambda_{ik} \lambda_{ij}^2 = \lambda_{jk}$$

We know that if we do a pair switching twice, you end up back where you started, which tells you $\lambda_{ij}^2 = 1$, so $\lambda_{ij} = \lambda_{ik}$. 

(c) [3] Argue that for any pair $\lambda_{ij}$ and $\lambda_{kl}$, $\lambda_{ij} = \lambda_{kl}$, whether there are any matches or not. Hence there is only one common eigenvalue for all pair switchings.

The order on the subscripts doesn’t matter, $\lambda_{ij} = \lambda_{ji}$ since they represent the same pair switching. There are three cases: no indices match, one index matches, and both indices match. If both match, we have $\lambda_{ij} = \lambda_{ij}$, which is trivial. If one matches, we have $\lambda_{ik} = \lambda_{jk}$ from part (b). If neither matches, then we can argue that $\lambda_{ij} = \lambda_{kl} = \lambda_{kl}$, since the intermediate one matches one index in each case. So we’re done!