Solutions to Chapter 11

1. [5] An electron in an unknown spin state $|a\rangle$ is brought into proximity with a second electron in a known spin state $|b\rangle$. We wish to make the spin of the second electron match the first. A quantum Xerox device will copy it onto the second spin, so $U_{\text{Xerox}} |a,b\rangle = |a,a\rangle$. A quantum teleporter will swap the two spin states, as $U_{\text{Teleport}} |a,b\rangle = |b,a\rangle$.

   (a) [3] By considering the three initial spin states $|a\rangle = |+\rangle$, $|a\rangle = |-\rangle$, and $|a\rangle = \frac{-1}{\sqrt{2}} (|+\rangle + |-\rangle)$, show that the quantum Xerox device is impossible.

   If the quantum Xerox device exists, it must change the state $|+,b\rangle$ into $|+,+\rangle$ and $|-,b\rangle$ into $|-,+\rangle$, in other words

   $$U_{\text{Xerox}} |+,b\rangle = |+,+\rangle \quad \text{and} \quad U_{\text{Xerox}} |-,b\rangle = |-,+\rangle$$

   However, $U_{\text{Xerox}}$ is a linear operator, and it follows that

   $$U_{\text{Xerox}} \left[ \frac{1}{\sqrt{2}} (|+,b\rangle + |-,b\rangle) \right] = \frac{1}{\sqrt{2}} (|+,+\rangle + |-,+\rangle).$$

   However, the quantum Xerox device is supposed to evolve this state into

   $$U_{\text{Xerox}} \left[ \frac{1}{\sqrt{2}} (|+,b\rangle + |-,b\rangle) \right] = \frac{1}{\sqrt{2}} (|+,+\rangle) \otimes \frac{1}{\sqrt{2}} (|+,+\rangle) = \frac{1}{2} (|+,+\rangle + |+,+\rangle + |-,+\rangle + |-,+\rangle)$$

   Obviously, these equations are inconsistent, and hence this is impossible.

   (b) [2] By considering the same three initial states, show that the same problem does not apparently occur for the quantum teleport device.

   The quantum teleport device should evolve the states according to

   $$U_{\text{Teleport}} |+,b\rangle = |b,+\rangle \quad \text{and} \quad U_{\text{Teleport}} |-,b\rangle = |b,-\rangle$$

   and therefore by linearity,

   $$U_{\text{Teleport}} \left[ \frac{1}{\sqrt{2}} (|+,b\rangle + |-,b\rangle) \right] = \frac{1}{\sqrt{2}} (|b,+\rangle + |b,-\rangle)$$

   But this is exactly what we would want it to do, so there is, in fact, no problem in this case.
2. \([10]\) At \(t = 0\), the wave function of a free particle is given by
\[
\Psi(x, t = 0) = (A/\pi)^{1/4} \exp\left[-\frac{i}{2} A(x - b)^2\right]
\]

Find \(\Psi(x, t)\), using the propagator.

This is straightforward, though boring. The answer is
\[
\Psi(x, t) = \int dx_0 K(x, t; x_0, 0) \Psi(x_0, 0)
\]
\[
= \left(\frac{A}{\pi}\right)^{1/4} \sqrt{\frac{m}{2\pi iht}} \int dx_0 \exp\left[\frac{im(x - x_0)^2}{2ht}\right] \exp\left[-\frac{1}{2} A(x_0 - b)^2\right]
\]
\[
= \left(\frac{A}{\pi}\right)^{1/4} \sqrt{\frac{m}{2\pi iht}} \exp\left[\frac{im}{A/2 - im/2ht}\right] \exp\left[-\frac{1}{4} \left(\frac{-imx/ht + Ab}{A/2 - im/2ht}\right)^2\right] \exp\left[-\frac{1}{2} A\frac{Ab^2}{2ht}\right]
\]
\[
= \left(\frac{A}{\pi}\right)^{1/4} \sqrt{\frac{m}{Ahti + m}} \exp\left[-\frac{1}{2} \left(\frac{A^2 b^2 h^2 t^2}{Ahti + m} - \frac{2Imxbht - m^2 x^2}{2(Ahti + m)ht} + \frac{imx^2 - hAb^2}{2ht}\right)\right]
\]
\[
= \left(\frac{A}{\pi}\right)^{1/4} \sqrt{\frac{m}{Ahti + m}} \exp\left[-\frac{1}{2} \left(\frac{Ab htim - 2Imxbht + Ahtx^2}{2(Ahti + m)ht}\right)\right]
\]
\[
= \left(\frac{A}{\pi}\right)^{1/4} \sqrt{\frac{m}{m + Ahti}} \exp\left[-\frac{1}{2} A\left(\frac{x-b}{m + Ahti}\right)^2\right] = \left(\frac{A}{\pi}\right)^{1/4} \sqrt{\frac{m}{m + Ahti}} \exp\left[-mA\left(\frac{x-b}{2(m + Ahti)}\right)^2\right]
\]

The final form makes it clear at least that if \(t = 0\), the wave function matches the initial wave function. Though it is less obvious, the wave function will spread out over time.