4. [15] An electron is in the + state when measured in the direction \( S_\theta = S_z \cos \theta + S_y \sin \theta \), so that \( S_\theta \left| +_\theta \right\rangle = +\frac{1}{\sqrt{2}}h \left| + \right\rangle \). However, the angle \( \theta \) is uncertain. In each part, it is probably a good idea to check at each step that the trace comes out correctly.

(a) [3] Suppose the angle is \( \theta = \pm \frac{\pi}{2} \), with equal probability for each angle. What is the state operator in the conventional \( \left| \pm \right\rangle \) basis?

We have, from a variety of sources, the states in terms of this basis, which is \( \left| +_\theta \right\rangle = \cos \left( \frac{\theta}{2} \right) \left| + \right\rangle + \sin \left( \frac{\theta}{2} \right) \left| - \right\rangle \). The state vector is then simply taken by averaging the results for the two angles, so

\[
\rho = \frac{1}{2} \sum_{\theta=\pm} \left| +_\theta \right\rangle \langle +_\theta \right| = \frac{1}{2} \left[ \begin{pmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & -\cos \frac{\theta}{2} \end{pmatrix} \left( \begin{pmatrix} \cos \frac{\theta}{2} & -\sin \frac{\theta}{2} \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \right) \right]
\]

\[
= \frac{1}{2} \left[ \begin{pmatrix} \cos^2 \frac{\theta}{2} & -\cos \frac{\theta}{2} \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \sin \frac{\theta}{2} & \sin^2 \frac{\theta}{2} \end{pmatrix} + \begin{pmatrix} \cos^2 \frac{\theta}{2} & \cos \frac{\theta}{2} \sin \frac{\theta}{2} \\ -\cos \frac{\theta}{2} \sin \frac{\theta}{2} & \sin^2 \frac{\theta}{2} \end{pmatrix} \right] = \begin{pmatrix} \cos^2 \frac{\theta}{2} & 0 \\ 0 & \sin^2 \frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}.
\]

The result has trace one, so it’s probably right.

(b) [4] Suppose the angle \( \theta \) is randomly distributed in the range \( -\frac{\pi}{4} < \theta < \frac{\pi}{4} \), with all angles equally likely. What is the state operator in the conventional \( \left| \pm \right\rangle \) basis?

Instead of adding two angles, we need to integrate over all angles, and divide by the range of angles, which is \( \pi \), so we have

\[
\rho = \frac{1}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left| +_\theta \right\rangle \langle +_\theta \right| d\theta = \frac{1}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left( \begin{pmatrix} \cos \frac{1}{2} \theta & \sin \frac{1}{2} \theta \\ \sin \frac{1}{2} \theta & -\cos \frac{1}{2} \theta \end{pmatrix} \right) \left( \begin{pmatrix} \cos \frac{1}{2} \theta & \sin \frac{1}{2} \theta \\ \sin \frac{1}{2} \theta & -\cos \frac{1}{2} \theta \end{pmatrix} \right) d\theta
\]

\[
= \frac{1}{\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \begin{pmatrix} \cos^2 \frac{1}{2} \theta & \cos \frac{1}{2} \theta \sin \frac{1}{2} \theta \\ \cos \frac{1}{2} \theta \sin \frac{1}{2} \theta & \sin^2 \frac{1}{2} \theta \end{pmatrix} d\theta = \frac{1}{2\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \begin{pmatrix} 1 + \cos \theta & \sin \theta \\ \sin \theta & 1 - \cos \theta \end{pmatrix} d\theta
\]

\[
= \frac{1}{2\pi} \begin{pmatrix} \theta + \sin \theta & -\cos \theta \\ -\cos \theta & \theta - \sin \theta \end{pmatrix} \frac{\pi}{2} = \frac{1}{2\pi} \begin{pmatrix} \frac{\pi}{2} + 1 & 0 \\ 0 & \frac{\pi}{2} - 1 \end{pmatrix} - \frac{1}{2\pi} \begin{pmatrix} \frac{\pi}{2} - 1 & 0 \\ 0 & \frac{\pi}{2} + 1 \end{pmatrix} = \begin{pmatrix} \frac{1 + \frac{\pi}{2}}{2} & 0 \\ 0 & \frac{1}{2} - \frac{1}{\pi} \end{pmatrix}.
\]

Once again, the trace is one, so it’s probably correct.

(c) [4] Suppose the angle \( \theta \) is randomly distributed in the range \( -\pi < \theta < \pi \), with all angles equally likely. What is the state operator in the conventional \( \left| \pm \right\rangle \) basis?
This is identical to the previous part, except the range is twice as big and of course the limits of integration change, so we have

\[
\rho = \frac{1}{2\pi} \int_{-\pi}^{\pi} |_{\theta}^{\theta} \left| +_{\theta} \right> \left< +_{\theta} \right| d\theta = \cdots = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left( \begin{array}{cc} 1 + \cos \theta & \sin \theta \\ \sin \theta & 1 - \cos \theta \end{array} \right) d\theta = \frac{1}{4\pi} \left( \begin{array}{cc} \theta + \sin \theta & -\cos \theta \\ -\cos \theta & \theta - \sin \theta \end{array} \right)_{\pi}^{\pi}
\]

\[
= \frac{1}{4\pi} \left[ \left( \begin{array}{cc} \pi & -1 \\ -1 & \pi \end{array} \right) - \left( \begin{array}{cc} -\pi & -1 \\ -1 & -\pi \end{array} \right) \right] = \left( \begin{array}{cc} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{array} \right)
\]

So this is a completely unpolarized state operator. It again has trace one.

(d) [4] In each of the cases listed above, what is the expectation value of \( S_z \)?

The expectation value can be found via

\[
\langle S_z \rangle = \text{Tr} \left( \rho S_z \right) = \frac{1}{2} \hbar \text{Tr} \left( \rho \sigma_z \right)
\]

In every case, this trace is easy to work out.

\[
\langle S_z \rangle_a = \frac{1}{2} \hbar \text{Tr} \left( \begin{array}{cc} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) = \frac{1}{2} \hbar \left( \frac{3}{4} - \frac{1}{4} \right) = \frac{1}{2} \hbar,
\]

\[
\langle S_z \rangle_b = \frac{1}{2} \hbar \text{Tr} \left( \begin{array}{cc} \frac{1}{2} + \frac{1}{\pi} & 0 \\ 0 & \frac{1}{2} - \frac{1}{\pi} \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) = \frac{1}{2} \hbar \left( \frac{1}{2} + \frac{1}{\pi} - \frac{1}{2} + \frac{1}{\pi} \right) = \frac{1}{\pi} \hbar.
\]

\[
\langle S_z \rangle_b = \frac{1}{2} \hbar \text{Tr} \left( \begin{array}{cc} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) = \frac{1}{2} \hbar \left( \frac{1}{2} - \frac{1}{2} \right) = 0.
\]

Perhaps not surprisingly, the first two cases have a positive expectation value, while the third vanishes. This is because in the first two cases, though the spin is random, it’s definitely at an angle that is closer to \( +z \) than \( -z \), but in the third it is equally likely at all angles.