3. An electron in the ground state of hydrogen has, in spherical coordinates, the wave function \( \psi (r, \theta, \phi) = Ne^{-r/a} \) where \( N \) and \( a \) are positive constants.

Determine the normalization constant \( N \) and the probability that a measurement of the position will yield \( r > a \). Don’t forget you are working in three dimensions!

In three dimensions, when working in spherical coordinates, the normalization condition is

\[
1 = \iiint |\psi(r, \theta, \phi)|^2 \, d^3 r = \int_0^\infty r^2 \, dr \int_0^\pi \, d\theta \int_0^{2\pi} \, d\phi |\psi(r)|^2
\]

In this case, the wave function depends only on \( r \), so the inner two integrals are trivial.

\[
1 = 4\pi N^2 \int_0^\infty r^2 e^{-2r/a} \, dr = 4\pi N^2 \left( -\frac{1}{2} r^2 a - \frac{1}{3} ra^2 - \frac{1}{4} a^3 \right) e^{-2r/a} \bigg|_0^\infty = \pi a^3 N^2
\]

\[ N = \frac{1}{\sqrt{\pi a^3}} \]

\[
P(r > a) = 4\pi N^2 \int_a^\infty r^2 e^{-2r/a} \, dr = \frac{4}{a^3} \left( -\frac{1}{2} r^2 a - \frac{1}{3} ra^2 - \frac{1}{4} a^3 \right) e^{-2r/a} \bigg|_a^\infty = 5e^{-2} = 67.67\% 
\]
4. [10] For each of the normalized wave functions given below, find the Fourier transform \( \tilde{\psi}(k) \), and check that it satisfies the normalization condition

\[ \int_{-\infty}^{\infty} |\tilde{\psi}(k)|^2 dk = 1. \]

(a) \[5\]

\( \psi(x) = \left(\frac{A}{\pi}\right)^{1/4} \exp\left(ikx - \frac{1}{2}Ax^2\right) \]

This turns into a Gaussian of the type you can find in Appendix A:

\[ \tilde{\psi}(k) = \left(\frac{A}{\pi}\right)^{1/4} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \exp\left(iKx - \frac{1}{2}Ax^2\right) \exp(-ikx) \]

\[ = \left(\frac{A}{\pi}\right)^{1/4} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \exp\left[i(K-k)x - \frac{1}{2}Ax^2\right] = \left(\frac{A}{\pi}\right)^{1/4} \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{A/2}} \exp\left\{ \frac{[i(K-k)]^2}{4(A/2)} \right\} \]

\[ = (\pi A)^{-1/4} \exp\left[-\frac{(K-k)^2}{2A}\right] \]

We check this by simply seeing if the normalization works out:

\[ \int_{-\infty}^{\infty} |\tilde{\psi}(k)|^2 dk = \frac{1}{\sqrt{\pi A}} \int_{-\infty}^{\infty} \exp\left[-\frac{(K-k)^2}{2A}\right] dk = \frac{1}{\sqrt{\pi A}} \int_{-\infty}^{\infty} e^{-k^2/4} dk = \frac{1}{\sqrt{\pi A}} \sqrt{\frac{\pi}{1/4}} = 1. \]

(b) \[5\]

\( \psi(x) = \sqrt{\alpha} \exp(-\alpha |x|) \)

I find absolute value problems easiest to deal with by dividing the integral into two pieces, then letting \( x \to -x \) on half of it. It is helpful to know \( \int_{0}^{\infty} e^{-ax} dx = 1/a \) if \( a \) has a real positive part.

\[ \tilde{\psi}(k) = \sqrt{\alpha} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-|x|} e^{-ikx} = \frac{\alpha}{\sqrt{2\pi}} \left( \int_{0}^{\infty} e^{-ax-ikx} dx + \int_{-\infty}^{0} e^{ax-ikx} dx \right) \]

\[ = \frac{\alpha}{\sqrt{2\pi}} \left( \int_{0}^{\infty} e^{-ax} dx \right) = \frac{\alpha}{\sqrt{2\pi}} \left( \frac{1}{\alpha+ik} + \frac{1}{\alpha-ik} \right) = \frac{\alpha}{\sqrt{2\pi}} \frac{2\alpha}{\alpha^2 + k^2}. \]

Once again, we check it, using the trig substitution \( k = \alpha \tan \theta \) to complete the integral.

\[ \int_{-\infty}^{\infty} |\tilde{\psi}(k)|^2 dk = \frac{4\alpha^3}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{(k^2 + \alpha^2)^2} = \frac{2\alpha^3}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\alpha \sec^2 \theta d\theta}{\left( \alpha^2 \tan^2 \theta + \alpha^2 \right)^2} = \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta d\theta = 1. \]