6.4 [15] A particle of mass $M$ in three dimensions has potential $V(X, Y, Z) = \frac{1}{4} A(X^2 + Y^2)^2$.

(a) [6] Show that this Hamiltonian has two continuous symmetries, and that they commute. Call the corresponding eigenvalues $m$ and $k$. Are there any restrictions on $k$ and $m$?

First, it is obvious that the potential is independent of $Z$, and therefore there is a continuous translation symmetry in this direction. Secondly, it is easy to see that rotation about the $z$-axis leaves the Hamiltonian unchanged. Specifically, define a set of rotated operators

\[
X' = X \cos \theta - Y \sin \theta, \\
Y' = X \sin \theta + Y \cos \theta.
\]

Then if we treat the potential as $V(x, y) = \frac{1}{4} A(x^2 + y^2)^2$, then we have

\[
V(X', Y') = \frac{1}{4} A \left[ (X \cos \theta - Y \sin \theta)^2 + (X \sin \theta + Y \cos \theta)^2 \right]^2 \\
= \frac{1}{4} A \left[ X^2 \cos^2 \theta - 2XY \cos \theta \sin \theta + Y^2 \sin^2 \theta + X^2 \sin^2 \theta + 2XY \sin \theta \cos \theta + Y^2 \cos^2 \theta \right] \left[ X^2 \cos^2 \theta - 2XY \cos \theta \sin \theta + Y^2 \sin^2 \theta + X^2 \sin^2 \theta + 2XY \sin \theta \cos \theta + Y^2 \cos^2 \theta \right] \\
= \frac{1}{4} A \left( X^2 + Y^2 \right)^2 = V(X, Y).
\]

Because we have translation symmetry in the $z$-direction and rotation about the $z$-axis, our Hamiltonian will commute with the generators of these groups, $P_z$ and $L_z$. Our energy eigenstates can also be chosen to be eigenstates of these operators, and we will have

\[
P_z |\phi\rangle = \hbar k |\phi\rangle \quad \text{and} \quad L_z |\phi\rangle = \hbar m |\phi\rangle.
\]

As argued in class, the eigenvalue $m$ is forced to be an integer, though $k$ is unrestricted.

(b) [9] What would be an appropriate set of coordinates for writing the eigenstates of this Hamiltonian? Write the eigenstates as a product of three functions (which I call $Z$, $R$, and $\Phi$), and give me the explicit form of two of these functions.

Clearly, $z$ is a good coordinate to use, since our eigenstates of the Hamiltonian are eigenstates of $P_z$. However, since they are also eigenstates of $L_z$, it seems like a good idea to change coordinates to cylindrical coordinates $(\rho, \phi, z)$, which are related to Cartesian coordinates by

\[
\begin{align*}
x &= \rho \cos \phi \\
y &= \rho \sin \phi \\
z &= z
\end{align*}
\]

OR

\[
\begin{align*}
\rho &= \sqrt{x^2 + y^2} \\
\phi &= \tan^{-1}(y/x) \\
z &= z
\end{align*}
\]

If we write our wave function in terms of these coordinates, and assume it factors, we have
\[ \psi(\rho, \phi, z) = R(\rho)\Phi(\phi)Z(z) \]

If we demand that this be an eigenstate of \( P_z \) with eigenvalue \( \hbar k \), then we find

\[ \hbar kZ(z) = P_zZ(z) = \frac{\hbar}{i} \frac{\partial}{\partial z} Z(z) \quad \text{so that} \quad Z(z) = e^{iz} \cdot \]

Similarly, if we demand that \( \psi(\rho, \phi, z) \) be an eigenstate of \( L_z \) with eigenvalue \( \hbar m \), then we find

\[ \hbar m\Phi(\phi) = L_z\Phi(\phi) = \frac{\hbar}{i} \frac{\partial}{\partial \phi} \Phi(\phi) \quad \text{so that} \quad \Phi(\phi) = e^{im\phi} \cdot \]

There is a certain arbitrariness in normalization, and the choices we have made have perhaps not been the best, but up to a constant, we therefore find

\[ \psi(\rho, \phi, z) = R(\rho)e^{iz+im\phi} \cdot \]

If we wished, we could now easily write an explicit equation for the radial function \( R \).

Writing the Laplacian that is implicit in the kinetic term in the Hamiltonian in cylindrical coordinates, we find

\[
\begin{aligned}
H\psi &= -\frac{\hbar^2}{2M} \left( \frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} \right) + \frac{1}{4} A(\rho^2)^2 \psi.
\end{aligned}
\]

Plugging in our explicit form for the wave function, and using Schrödinger’s equation \( H\psi = E\psi \), we have

\[
\begin{aligned}
ER &= -\frac{\hbar^2}{2M} \left( \frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} \right) + \left[ \frac{\hbar^2 k^2}{2M} + \frac{\hbar^2 m^2}{2M \rho^2} + \frac{1}{4} A\rho^4 \right] R.
\end{aligned}
\]