2. [10] Suppose that \( L \) and \( S \) are two sets of angular momentum-like operators that commute with each other, so that \([L_i, S_j] = 0\). In this problem, you may assume from the commutation relations that it follows that \([L^2, L] = [S^2, S] = 0\).

(a) [2] Define \( J = L + S \). Show that \( J \) is also an angular momentum-like operator. It follows automatically that \([J^2, J] = 0\).

We simply have to work out the commutator of each component of \( J \) with each other. We’ll take advantage of the Levi-Civita tensor to show that
\[
\begin{align*}
[j_i, j_j] &= [l_i + S_i, l_j + S_j] = [l_i, l_j] + [S_i, S_j] = i\hbar \sum_k \epsilon_{ijk} l_k + i\hbar \sum_k \epsilon_{ijk} S_k = i\hbar \sum_k \epsilon_{ijk} j_k.
\end{align*}
\]

This saved us the work of doing it three times.

(b) [2] Show that \([L^2, J] = [S^2, J] = 0\).

This is really pretty trivial.

\[
\begin{align*}
[L^2, J] &= [L^2, L] + [L^2, S] = 0 + 0 = 0, \\
[S^2, J] &= [S^2, L] + [S^2, S] = 0 + 0 = 0.
\end{align*}
\]

(c) [3] Convince yourself (and me) that the four operators \( J^2, L^2, S^2 \), and \( J_z \) all commute with each other (this is six commutators in all).

Since \( L^2 \) and \( S^2 \) commute with \( J \), it follows automatically that they commute with \( J^2 \) and \( J_z \) (that’s four commutators so far). Since all the \( L \)’s and \( S \)’s commute with each other, it follows that \( L^2 \) and \( S^2 \) commute with each other. Finally, as mentioned in part (a), since \([J^2, J] = 0\), \( J^2 \) commutes with \( J_z \).

(d) [3] Convince yourself that \( L_z \) and \( S_z \) do not commute with \( J^2 \).

We simply try to do the commutation relations and see if it works.

\[
\begin{align*}
[J^2, L_z] &= [(L + S)^2, L_z] = [L^2 + 2L \cdot S + S^2, L_z] = [L^2, L_z] + 2[L \cdot S, L_z] + [S^2, L_z] \\
&= 2[L_z, L_z]S_x + 2[L_z, L_z]S_y + 2[L_z, L_z]S_z = -2i\hbar L_x S_x + 2i\hbar L_y S_y, \\
[J^2, S_z] &= [(L + S)^2, S_z] = [L^2 + 2L \cdot S + S^2, S_z] = [L^2, S_z] + 2[L \cdot S, S_z] + [S^2, S_z] \\
&= 2L_z [S_x, S_z] + 2L_y [S_y, S_z] + 2L_z [S_z, S_z] = -2i\hbar L_x S_y + 2i\hbar L_y S_x.
\end{align*}
\]

Not surprisingly, the sum of these two expressions is zero, which follows from \([J^2, J] = 0\).