3. [25] Suppose an electron lies in a region with electric and magnetic fields: \( \mathbf{B} = B\hat{z} \) and 
\( \mathbf{E} = \frac{m\omega_0^2}{e}x\hat{x} \).

(a) [2] Find the electric potential \( U(x) \) such that 
\( \mathbf{E} = -\nabla U(x) \) that could lead to this electric field.

We need the potential to get the derivative in the \( x \)-direction to yield \( -m\omega_0^2x/e \), which tells us that the correct choice is 
\( U(x) = -\frac{m\omega_0^2}{2e}x^2 \). This is easily checked.

(b) [3] The magnetic field is independent of translations in all three dimensions. However, the electrostatic potential is independent of translations in only two of those dimensions. Find a vector potential \( \mathbf{A} \) with 
\( \mathbf{B} = \nabla \times \mathbf{A} \) which has translation symmetry in the same two directions.

There are always multiple ways to choose to write the vector potential. The electric potential is translation invariant in the \( y \)- and \( z \)-directions, so it makes a lot of sense to try to make our vector potential independent of these two coordinates as well. This means when we write 
\( \mathbf{B} = \nabla \times \mathbf{A} \), we’re going to need to get the magnetic field from taking derivatives in the \( x \)-direction. The way the curl works, this will work out if we choose the magnetic field to lie in the \( y \)-direction, and it isn’t hard to see that this works if 
\( \mathbf{A} = Bx\hat{y} \).

(c) [4] Write out the Hamiltonian for this system. Eliminate \( B \) in terms of the cyclotron frequency \( \omega_B = eB/m \). What two translation operators commute with this Hamiltonian? What spin operator commutes with this Hamiltonian?

The Hamiltonian is
\[
H = \frac{1}{2m} \left( \mathbf{P} + e\mathbf{A} \right)^2 - eU + \frac{ge}{2m} \mathbf{B} \cdot \mathbf{S} = \frac{1}{2m} \left[ P_x^2 + \left( P_y + eBX \right)^2 + P_z^2 \right] + \frac{1}{2} m\omega_0^2 X^2 + \frac{ge}{2m} B S_z \\
= \frac{1}{2m} \left[ P_x^2 + \left( P_y + m\omega_B X \right)^2 + P_z^2 \right] + \frac{1}{2} m\omega_0^2 X^2 + \frac{1}{2} g \omega_B S_z
\]

This commutes with \( P_y \), \( P_z \), and \( S_z \). Life is good.

(d) [3] Write your wave function in the form \( \psi(\mathbf{r}) = X(x)Y(y)Z(z)|m_z\rangle \). Based on some of the operators you worked out in part (c), deduce the form of two of the unknown functions.

Since our wave function commutes with \( P_y \) and \( P_z \), we can choose it to be eigenstates of two of these operators, and consequently they will look like 
\( Y(y) = e^{ik_y y} \) and 
\( Z(z) = e^{ik_z z} \).

These will have eigenvalues \( \hbar k_y \) and \( \hbar k_z \) under these two operators.
(e) [3] Replace the various operators by their eigenvalues in the Hamiltonian. The non-constant terms should be identifiable as a shifted harmonic oscillator.

Replacing the operators by their eigenvalues, the Hamiltonian becomes

$$H = \frac{1}{2m} \left[ P_x^2 + \left( \hbar \omega_y + m \omega_y X \right)^2 + \hbar^2 k_z^2 \right] + \frac{1}{2} m \omega_x X^2 + \frac{1}{2} \hbar \hbar \omega_y m_x,$$

$$= \frac{P_x^2}{2m} + \frac{1}{2} m \left( \omega_y^2 + \omega_x^2 \right) X^2 + \hbar \omega_y X + \frac{\hbar^2 k_y^2}{2m} + \frac{\hbar^2 k_z^2}{2m} + \frac{1}{2} \hbar \hbar \omega_y m_x,$$

The last few terms are constants, and the rest is simply a shifted harmonic oscillator.

(f) [4] Make a simple coordinate replacement that shifts it back. If your formulas match mine up to now, they should look like $X = X' - \hbar \omega_y / \left[ m \left( \omega_y^2 + \omega_x^2 \right) \right].$

We try the suggested substitution.

$$H = \frac{P_x^2}{2m} + \frac{1}{2} m \left( \omega_y^2 + \omega_x^2 \right) \left[ X' - \frac{\hbar \omega_y}{m \left( \omega_y^2 + \omega_x^2 \right)} \right]^2 + \hbar \omega_y \left[ X' - \frac{\hbar \omega_y}{m \left( \omega_y^2 + \omega_x^2 \right)} \right] + \frac{\hbar^2 (k_y^2 + k_z^2)}{2m},$$

$$= \frac{P_x^2}{2m} + \frac{1}{2} m \left( \omega_y^2 + \omega_x^2 \right) X'^2 - \frac{\hbar^2 k_y^2 \omega_x^2}{2m \left( \omega_y^2 + \omega_x^2 \right)} + \frac{\hbar^2 k_x^2}{2m} + \frac{\hbar^2 k_z^2}{2m} + \frac{1}{2} \hbar \hbar \omega_y m_x.$$

(g) [3] Find the energies of the Hamiltonian

The first two terms are simply a Harmonic oscillator, now not shifted, and the energies are just $\hbar \omega \left( n + \frac{1}{2} \right),$ where $\omega = \sqrt{\omega_y^2 + \omega_x^2}.$ Therefore the energies are in total

$$E = \hbar \omega \left( n + \frac{1}{2} \right) + \frac{\hbar^2 k_y^2}{2m} + \frac{\hbar^2 k_x^2 \omega_x^2}{2m \left( \omega_y^2 + \omega_x^2 \right)} + \frac{1}{2} \hbar \hbar \omega_y m_x.$$

(h) [3] Check that they give sensible answers in the two limits when there is no electric field (pure Landau levels) or no magnetic fields (pure harmonic oscillator plus y- and z-motion).

If there are no electric fields, then $\omega_y = 0,$ and we have

$$E = \hbar^2 k_y^2 / 2m + \hbar \omega_y \left( n + \frac{1}{2} + \frac{1}{2} \hbar \hbar m_x \right).$$ This is exactly what we would expect. If there are no magnetic fields, then $\omega_x = 0,$ and we have $E = \hbar \omega_y \left( n + \frac{1}{2} \right) + \hbar^2 \left( k_z^2 + k_y^2 \right) / 2m,$ which is a harmonic oscillator added to motion in the y- and z-direction.