1. **[15 points]** Consider the wave function \( \psi(x) = N \exp\left(-\frac{1}{2}ax^2 + i\beta x\right) \).
   
   (a) [6] What is the correct normalization \( N \)?
   
   (b) [9] What is \( \langle X \rangle \) and \( \langle P \rangle \) for this state?

2. **[30 points]** In a three-dimensional space with orthonormal basis \( \{|x\rangle, |y\rangle, |z\rangle\} \), the operator \( L \) has the following properties:
   
   \[
   L|x\rangle = i|y\rangle, \quad L|y\rangle = -i|x\rangle, \quad L|z\rangle = 0.
   \]
   
   (a) [5] Write \( L \) as a 3x3 matrix. Is it Hermitian?
   
   (b) [12] Find the eigenvalues and orthonormal eigenvectors of \( L \).
   
   (c) [7] A system is in the state \( |x\rangle \) when \( L \) is measured. What are the possible outcomes of that measurement, and the corresponding probabilities?
   
   (d) [6] For each of the outcomes in part (c), what will the state vector look like afterwards?

3. **[10 points]** Consider the harmonic oscillator with mass \( m \) and angular frequency \( \omega \). For the quantum state
   
   \[
   \psi_3^4 = \frac{1}{\sqrt{5}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle,
   \]
   
   find \( X \) and \( P \).

4. **[20 points]** A particle of mass \( m \) lies in the infinite square well with allowed region \( 0 < x < a \).
   
   At \( t = 0 \), the wave function takes the form \( \Psi(x,t=0) = \frac{1}{\sqrt{a}} \) in the allowed region and it vanishes elsewhere.
   
   (a) [8] Write this state in the form \( \Psi(0) = \sum_n c_n |\phi_n\rangle \), where \( |\phi_n\rangle \) is the \( n \)th eigenstate of the Hamiltonian. Work out a formula for the \( c_n \)’s.
   
   (b) [5] Check that in the new basis, \( |\Psi(0)\rangle \) is properly normalized. For this purpose, you can just keep a couple non-zero terms and make sure the sum is coming out about right.
   
   (c) [7] Write \( \Psi(x,t) \) as a function of time in terms of the eigenstate basis, and \( \Psi(x,t) \). You are not expected to do the sums.

5. **[10 points]** The angular momentum operator \( L_z \) has the commutation properties
   
   \[
   [L_z, X] = i\hbar Y \quad \text{and} \quad [L_z, Y] = -i\hbar X.
   \]
   
   Define two new operators \( A = \frac{1}{2} X^2 - \frac{1}{2} Y^2 \) and \( B = XY \).
   
   (a) [6] Prove the following true statements: \( [L_z, A] = 2i\hbar B \) and \( [L_z, B] = -2i\hbar A \)
   
   (b) [4] Based on the commutation relations found in (a), write two uncertainty relations.
6. **[15 points]** A particle of mass \( \mu \) in two dimensions lies in the potential \( V = A \sqrt{X^2 + Y^2} \).

(a) **[5]** What continuous symmetry and corresponding generator commutes with this Hamiltonian? What are the corresponding possible eigenvalues of this generator? Are there any restrictions on this eigenvalue?

(b) **[5]** Write the wave function as \( \psi = \psi(\rho, \phi) = R(\rho) \Phi(\phi) \) in polar coordinates. Write one of these explicitly.

(c) **[5]** Deduce an ordinary differential equation for the other one. The 2D Laplacian in polar coordinates is given below. Do not attempt to solve this equation.

### Possibly Helpful Formulas:

#### Infinite Square Well
\[
\psi_n(x) = \sqrt{\frac{2}{a}} \sin \left( \frac{n \pi x}{a} \right) \\
E_n = \frac{\pi^2 \hbar^2 n^2}{2ma^2}
\]

#### Matrices for Operators
\[
A = \begin{pmatrix}
\langle \phi_1 | A | \phi_1 \rangle & \langle \phi_1 | A | \phi_2 \rangle & \cdots \\
\langle \phi_2 | A | \phi_1 \rangle & \langle \phi_2 | A | \phi_2 \rangle & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\]

#### Generators of Symmetries
\[
T(a) = \exp(-i a \cdot P/\hbar) \\
R(\mathbf{R}, \theta) = \exp(-i \theta \hat{\mathbf{n}} \cdot \mathbf{L}/\hbar)
\]

#### Rotation Matrices 2D
\[
\mathbf{R}(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\
\sin \theta & \cos \theta \end{pmatrix}
\]

#### Commutators
\[
\]

#### 2D Laplacian
\[
\nabla^2 \psi = \frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2}
\]

### Possibly Helpful Integrals:

**Definite Integrals:**

\[
\int_{-\infty}^{\infty} e^{-Ax^2} \, dx = \frac{\pi}{\sqrt{A}}, \quad \int_{-\infty}^{\infty} e^{-Ax^2+2Bx} \, dx = \frac{\pi}{\sqrt{A}} e^{B^2/(4A)} \\
\int_{-\infty}^{\infty} xe^{-Ax^2} \, dx = 0, \quad \int_{-\infty}^{\infty} xe^{-Ax^2+2Bx} \, dx = \frac{B}{2A} \sqrt{\frac{\pi}{A}} e^{-B^2/(4A)} \\
\int_{-\infty}^{\infty} x^2 e^{-Ax^2} \, dx = \frac{1}{2A} \sqrt{\frac{\pi}{A}}, \quad \int_{-\infty}^{\infty} x^2 e^{-Ax^2+2Bx} \, dx = \left( \frac{1}{2A} + \frac{B^2}{4A^2} \right) \sqrt{\frac{\pi}{A}} e^{-B^2/(4A)}
\]

**Indefinite Integrals:** Note that in each case, a constant + C is understood to be added at the end.

\[
\int \sin(Ax) \, dx = -\frac{1}{A} \cos(Ax), \quad \int \cos(Ax) \, dx = \frac{1}{A} \sin(Ax) \\
\int \sin^2(Ax) \, dx = \frac{x}{2} - \frac{1}{4A} \sin(2Ax), \quad \int \cos^2(Ax) \, dx = \frac{x}{2} + \frac{1}{4A} \sin(2Ax) \\
\int \sin(Ax) \cos(Ax) \, dx = -\frac{1}{4A} \cos(2Ax)
\]