1. **[20 points]** Consider the wave function \( \psi(x) = N(x + ia)^2 \). This state, once properly normalized, has expectation values \( \langle P \rangle = \frac{\hbar}{2} a^{-1} \) and \( \langle P^2 \rangle = 3\hbar^2 a^{-2} \).
   (a) [7] What is the correct normalization \( N \)?
   (b) [7] What is \( \langle X \rangle \) and \( \langle X^2 \rangle \) for this state?
   (c) [6] Find the uncertainties \( \Delta X \) and \( \Delta P \) and show that they satisfy the uncertainty relation.

2. **[20 points]** A particle of mass \( m \) lies in the infinite square well with allowed region \( 0 < x < a \).
   At \( t = 0 \), the wave function takes the form
   \[
   \Psi(x, t = 0) = N \sin\left(\frac{2\pi x}{a}\right) \cos\left(\frac{5\pi x}{a}\right)
   \]
   in the allowed region and it vanishes elsewhere.
   (a) [7] Write this state in the form \( |\psi(0)\rangle = \sum_n c_n |\phi_n\rangle \). Some helpful integrals are provided.
   (b) [6] Determine the normalization constant \( N \) such that \( |\Psi\rangle \) is normalized.
   (c) [7] Write \( |\psi(t)\rangle \) as a function of time in terms of the eigenstate basis, and write \( \Psi(x, t) \).

3. **[25 points]** In a certain basis, the Hamiltonian takes the form
   \[
   H = \hbar \omega \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}.
   \]
   (a) [12] Find the eigenvalues and normalized eigenvectors of this Hamiltonian.
   (b) [7] At \( t = 0 \), the state is in the state \( |\Psi(t = 0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \). Find \( |\Psi(t)\rangle \) at all times.
   (c) [6] At time \( t \), if we were to measure the energy, what would be the possible outcomes and corresponding probabilities?

4. **[15 points]** Consider the harmonic oscillator with mass \( m \) and angular frequency \( \omega \).
   (a) [7] For which non-negative integers \( q \) will the matrix elements \( \langle 47 | X^q | 50 \rangle \) or \( \langle 47 | P^q | 50 \rangle \) be non-zero? I want a complete rule that lets me tell when they are non-zero.
   (b) [8] For the smallest \( q \) for which they do not vanish, compute them.
5. [20 points] Bottomonium consists of a bottom quark and bottom anti-quark, each of mass $m_b$, bound by a potential that is approximately $V(r) = Ar$, where $r$ is the distance between them.

(a) [4] Find a formula for the reduced mass of this system in terms of the quark mass $m_b$.

(b) [5] What is an appropriate choice of coordinates for this system? Name two operators that commute with each other and with the Hamiltonian. Label the eigenstates of $H$ by their eigenvalues under these two new operators. What can you tell me about these eigenvalues?

(c) [5] Factor the wave function into an angular and a radial part. Describe completely one of these functions.

(d) [6] For the remaining function, write an ordinary differential equation for the function. Do not attempt to solve it.

Possibly Helpful Formulas:

<table>
<thead>
<tr>
<th>Generators of Symmetries</th>
<th>Harmonic Oscillator</th>
<th>$P^2$ in Spherical or Cylindrical Coords.</th>
</tr>
</thead>
</table>
| $T(a) = \exp(-i a \cdot \mathbf{P}/\hbar)$ | $X = \frac{\hbar}{2m\omega}(a + a^\dagger)$ | $P^2\psi = -\frac{\hbar^2}{2m}
\frac{\partial^2}{\partial \rho^2}(\rho \frac{\partial \psi}{\partial \rho}) + \frac{1}{\rho^2}L^2\psi + P_z^2\psi$ |
| $R(\mathbf{R}(\hat{n}, \theta)) = \exp(-i \theta \hat{n} \cdot \mathbf{L}/\hbar)$ | $P = i\sqrt{\frac{2\hbar m\omega}{\pi}}(a^\dagger - a)$ | $a|n\rangle = \sqrt{n}|n-1\rangle$ |
| | $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ | $P_z^2\psi = \frac{\hbar^2}{r} \frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2}L^2\psi$ |

Possibly Helpful Integrals:

Definite Integrals: $n$, $m$ and $p$ are assumed to be positive integers

\[
\int_0^a \sin \left( \frac{n\pi x}{a} \right) \sin \left( \frac{m\pi x}{a} \right) \, dx = \frac{a}{2} \delta_{nm} \\
\int_0^a \cos \left( \frac{n\pi x}{a} \right) \sin \left( \frac{m\pi x}{a} \right) \cos \left( \frac{p\pi x}{a} \right) \, dx = \frac{a}{2} \left( 2 \delta_{n,m+p} + \delta_{n,m+p} - \delta_{n,m+p} \right) \\
\int_0^a \cos \left( \frac{n\pi x}{a} \right) \cos \left( \frac{m\pi x}{a} \right) \cos \left( \frac{p\pi x}{a} \right) \, dx = \frac{a}{2} \left( 2 \delta_{n,m+p} + 2 \delta_{n,m+p} + \delta_{n,m+p} \right)
\]

\[
\int_{-\infty}^\infty \frac{x^n \, dx}{(x^2 + b^2)^m} = \begin{cases} 
\frac{\pi}{2b^m} & \text{if } n = 0, \\
\frac{\pi}{2b^m} & \text{if } n = m, \\
\frac{\pi}{b} & \text{if } n \geq m 
\end{cases}
\]

\[
\int_{-\infty}^\infty \frac{x \, dx}{(x^2 + b^2)^2} = \frac{\pi}{2b^3}, \quad \int_{-\infty}^\infty \frac{x^2 \, dx}{(x^2 + b^2)^2} = \frac{\pi}{2b}, \quad \int_{-\infty}^\infty \frac{x \, dx}{(x^2 + b^2)^4} = \frac{5\pi}{16b^7}, \quad \int_{-\infty}^\infty \frac{x^2 \, dx}{(x^2 + b^2)^4} = \frac{\pi}{16b^5} 
\]

Indefinite Integral: \[\int (x + ib)^n \, dx = \frac{1}{n+1} (x + ib)^{n+1} + C\]