1. Consider the wave function \( \psi(x) = \begin{cases} Nx(a-x) & 0 < x < a, \\ 0 & \text{otherwise.} \end{cases} \)

Once properly normalized, this wave function has \( \langle X \rangle = \frac{1}{2}a \) and \( \langle X^2 \rangle = \frac{7}{8}a^2 \).

(a) [5] What is the correct normalization \( N \)?

We insist that the normalization integral yields one, so we have
\[
1 = \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = N^2 \int_0^a \left[ x(a-x) \right]^2 \, dx = N^2 \int_0^a \left( a^2 x^2 - 2ax^3 + x^4 \right) \, dx \\
= N^2 \left( \frac{1}{4} a^2 x^3 - \frac{1}{2} ax^4 + \frac{1}{2} x^5 \right)_0^a = N^2 a^5 \left( \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \right) = \frac{1}{30} N^2 a^5, \\
N = \sqrt{\frac{30}{a^5}}.
\]

(b) [8] What are \( \langle P \rangle \) and \( \langle P^2 \rangle \) for this state?

We simply insert the operator \( P = -i \frac{d}{dx} \) to find
\[
\langle P \rangle = -i \hbar \int_{-\infty}^{\infty} \psi^*(x) \frac{d}{dx} \psi(x) \, dx = N^2 \int_0^a \left[ x(a-x) \right] \frac{d}{dx} \left[ x(a-x) \right] \, dx \\
= -i \hbar \frac{30}{a^3} \int_0^a (ax-x^2)(a-2x) \, dx = -i \hbar \frac{30}{a^3} \int_0^a \left( a^2 x^3 - 3ax^2 + 2x^4 \right) \, dx \\
= -i \hbar \frac{30}{a^3} \left( \frac{1}{2} a^2 x^2 - ax^3 + \frac{1}{2} x^4 \right)_0^a = -i \hbar \frac{30}{a} \left( \frac{1}{2} - 1 + \frac{1}{2} \right) = 0,
\]
\[
\langle P^2 \rangle = -\hbar^2 \int_{-\infty}^{\infty} \psi^*(x) \frac{d^2}{dx^2} \psi(x) \, dx = -\hbar^2 \frac{30}{a^3} \int_0^a \left[ x(a-x) \right] \frac{d^2}{dx^2} \left[ x(a-x) \right] \, dx \\
= -60 \hbar^2 \frac{a}{a^3} \int_0^a (ax-x^2) \, dx = \frac{60}{a^5} \left( \frac{1}{2} ax^2 - \frac{1}{2} x^3 \right)_0^a = \frac{60}{a^5} \left( \frac{1}{2} - \frac{1}{4} \right) = \frac{10}{a^2} \hbar^2.
\]

(c) [7] Find the uncertainties \( \Delta x \) and \( \Delta p \) and show that they satisfy the uncertainty relation.

\[
\Delta x = \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = \sqrt{\frac{7}{8}a^2 - \left( \frac{1}{2}a \right)^2} = a \sqrt{\frac{7}{8} - \frac{1}{4}} = a \sqrt{\frac{3}{8}} = \frac{a \sqrt{6}}{28},
\]
\[
\Delta p = \sqrt{\langle P^2 \rangle - \langle P \rangle^2} = \sqrt{\frac{10 a^2 \hbar^2}{a^2} - 0^2} = \frac{\sqrt{10} \hbar}{a}.
\]

This yields \( (\Delta x)(\Delta p) = \sqrt{\frac{15}{4}} \hbar = 0.578 \hbar > \frac{1}{2} \hbar \).
2. A particle of mass \(m\) lies in the infinite square well with allowed region \(0 < x < a\). The wave function takes the form \(\psi(x) = \begin{cases} N \sin^2\left(\frac{\pi x}{a}\right) & 0 < x < a, \\ 0 & \text{elsewhere}. \end{cases} \)

(a) [5] Determine the normalization constant \(N\).

With the help of the helpful integrals, we have

\[
1 = \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = N^2 \int_{0}^{a} \sin^4\left(\frac{\pi x}{a}\right) \, dx = N^2 \int_{0}^{a} \sin^2\left(\frac{\pi x}{a}\right) \sin^2\left(\frac{\pi x}{a}\right) \, dx = \frac{3}{4} N^2 a, \\
N = \sqrt{8/3a}.
\]

(b) [7] Write this state in the form \(|\psi\rangle = \sum_n c_n |\phi_n\rangle\), where \(|\phi_n\rangle\) are the energy eigenstates. Some helpful integrals are provided.

The normalized energy eigenstates and eigenvalues are given by

\[
\phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi nx}{a}\right), \quad E_n = \frac{\pi^2 n^2 h^2}{2ma^2}.
\]

The overlaps \(c_n\) are given by

\[
c_n = \langle \phi_n | \psi \rangle = \sqrt{\frac{2}{3a}} \int_{-a}^{a} \sin\left(\frac{\pi nx}{a}\right) \sin\left(\frac{\pi x}{a}\right) \, dx = \frac{4 \cdot (1)^2 a}{\sqrt{3a \pi n} (4 - n^2)} \quad \text{if } n \text{ odd, zero otherwise.}
\]

Simplifying and substituting into the sum, we have

\[
|\psi\rangle = \sum_{n \text{ odd}} \frac{16}{\pi n \sqrt{3} (4 - n^2)} |\phi_n\rangle.
\]

(c) [8] If we were to measure the energy, what would be the possible outcomes and corresponding probabilities? Give a general formula, and find the numeric value as a percentage for the first three non-zero outcomes.

The energies were given above, namely \(E_n = \frac{\pi^2 n^2 h^2}{2ma^2}\), but the probability vanishes unless \(n\) is odd. For \(n\) odd, we have

\[
P(n) = |\langle \phi_n | \psi \rangle|^2 = |c_n|^2 = \frac{256}{3\pi^2 n^2 \left(n^2 - 4\right)^2}.
\]

The table at right gives the resulting probabilities for the first three non-zero cases. Note that the probabilities add to 99.99%. They should total one, which doubtless just represents the contribution from larger \(n\).
3. Consider the harmonic oscillator with mass \(m\) and angular frequency \(\omega\). At \(t = 0\), the system is in the state \(|\Psi(t = 0)\rangle = N \sum_{n=1}^{\infty} \frac{i^n}{n^2} |n\rangle\).

(a) [7] What is the correct normalization \(N\)? Some helpful sums are given on the next page.

We need to have

\[
1 = \langle \Psi | \Psi \rangle = N^2 \sum_{p=1}^{\infty} \frac{(-i)^p}{p^2} \langle p | \sum_{n=1}^{\infty} \frac{i^n}{n^2} |n\rangle = N^2 \sum_{p=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-i)^p}{p^2 n^2} \delta_{np} = N^2 \sum_{n=1}^{\infty} \frac{1}{n^4} = N^2 \zeta(4) = \frac{\pi^4 N^2}{90},
\]

\[
N = \sqrt{\frac{90}{\pi^2}}.
\]

(b) [5] Find the value of \(\langle P \rangle\) for this state. Simplify as much as possible.

We write the operators in terms of raising and lowering operators, so we have

\[
\langle P \rangle = \langle \Psi | P | \Psi \rangle = \frac{90}{\pi^4} i \sqrt{\frac{1}{2 \hbar \omega}} \left[ \sum_{p=1}^{\infty} \frac{(-i)^p}{p^2} \langle p | \sum_{n=1}^{\infty} \frac{i^n}{n^2} |n\rangle \left( a^\dagger - a \right) \left[ \sum_{n=1}^{\infty} \frac{i^n}{n^2} |n\rangle \right] \right]
\]

\[
= \frac{45}{\pi^4} i \sqrt{2 \hbar \omega} \left[ \sum_{p=1}^{\infty} \frac{(-i)^p}{p^2} \langle p | \sum_{n=1}^{\infty} \frac{i^n}{n^2} \left( \sqrt{n+1} |n+1\rangle - \sqrt{n} |n-1\rangle \right) \right]
\]

The smart way to simplify this is to use the delta function to do the \(p\)-sum on the first term and to do the \(n\) – sum on the second term. Then we have

\[
\langle P \rangle = \frac{45}{\pi^4} i \sqrt{2 \hbar \omega} \left[ \sum_{n=1}^{\infty} \frac{i^n}{n^2 (n+1)^2} \sqrt{n+1} - \sum_{p=1}^{\infty} \frac{i^{p+1}}{p^2 (p+1)^2} \sqrt{p+1} \right]
\]

\[
= \frac{45}{\pi^4} \sqrt{2 \hbar \omega} \left[ \sum_{n=1}^{\infty} \frac{1}{n^2 (n+1)^2} + \sum_{p=1}^{\infty} \frac{1}{p^2 (p+1)^2} \right] = \frac{90}{\pi^4} \sqrt{2 \hbar \omega} \sum_{n=1}^{\infty} \frac{1}{n^2 (n+1)^2}.
\]

Other than numerically, I don’t know of any way to simplify this further.

(c) [8] What is \(|\Psi(t)\rangle\) at all times?

Each of the eigenstates has energy \(E_n = \hbar \omega \left( n + \frac{1}{2} \right)\), so when we include time-dependance, they simply pick up a factor of \(\exp(-i E_n t / \hbar) = \exp\left[-i \left( n + \frac{1}{2} \right) \omega t \right]\). So the time state vector is

\[
|\Psi(t)\rangle = \sum_{n=1}^{\infty} c_n e^{-i \left( n + \frac{1}{2} \right) \omega t} |n\rangle = \frac{3 \sqrt{10}}{\pi^2} \sum_{n=1}^{\infty} \frac{i^n}{n^2} e^{-i \left( n + \frac{1}{2} \right) \omega t} |n\rangle.
\]
4. A hydrogen atom is in the state \( |n,l,m\rangle = |2,1,0\rangle \).

(a) [6] What would be the result if you measure the energy, orbital angular momentum squared \( L^2 \) and \( z \)-component \( L_z \)?

Because we are in an eigenstate of all three quantities, the three requested quantities are given by

\[
E = -\frac{13.6 \text{ eV}}{n^2} = -\frac{13.6 \text{ eV}}{2^2} = -3.40 \text{ eV},
\]

\[
L^2 = \hbar^2 (l^2 + l) = \hbar^2 (1^2 + 1) = 2\hbar^2,
\]

\[L_z = \hbar m = 0.\]

(b) [6] Write the explicit form of the wave function \( \psi(r,\theta,\phi) \).

We simply write it down using

\[
\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_l^m(\theta,\phi) = R_{21}(r)Y_1^0(\theta,\phi) = \frac{r e^{-r/2a}}{2\sqrt{6a^2}} \frac{\sqrt{3}}{2\sqrt{\pi}} \cos \theta = \frac{r e^{-r/2a}}{4\sqrt{2\pi a^5}} \cos \theta.
\]

(c) [8] Calculate the expectation value \( \langle R^{-1} \rangle \) for this wave function, where \( R \) is the distance from the origin operator.

We can save some steps using the fact that the spherical harmonics are orthonormal when integrated over angles, so we have

\[
\langle R^{-1} \rangle = \int \psi_{210}^*(r) r^{-1} \psi_{210}(r) d^3r = \int_0^\infty \left[ R_{21}(r) \right]^2 r^2 r^{-1} dr \int Y_1^0(\theta,\phi)^* Y_1^0(\theta,\phi) d\Omega
\]

\[
= \int_0^\infty \left[ \frac{r e^{-r/2a}}{2\sqrt{6a^5}} \right]^2 r dr = \frac{1}{24a^2} \int_0^\infty r^3 e^{-r/a} dr = \frac{1}{24a^5} a^4 3! = \frac{1}{4a}.
\]
5. In a certain basis, the state vector is given by $|\Psi\rangle = \left(\frac{1}{2} + \frac{2}{3}i\right)$, and the spin operator in the $x$-directions is given by $S_x = \frac{1}{2} \hbar \sigma_x = \frac{1}{2} \hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(a) [10] Find the eigenvalues and normalized eigenvectors of $S_x$.

We first find the eigenvalues and eigenvectors of the Pauli matrix $\sigma_x$, which are found from

$$0 = \det (\sigma_x - \lambda 1) = \det \begin{pmatrix} 0 - \lambda & 1 \\ 1 & \lambda \end{pmatrix} = \lambda^2 - 1,$$

$$\lambda = \pm 1.$$

The eigenvalues for $S_x$ will then be $\pm \frac{1}{2} \hbar$. We can then find the eigenvectors by giving them arbitrary components, and solving the eigenvector equation, so we have

$$\pm \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}, \quad \text{so} \quad \beta = \pm \alpha, \quad \left| \pm \frac{1}{2} \hbar \right\rangle = \begin{pmatrix} \alpha \\ \pm \alpha \end{pmatrix}.$$

Normalizing them, we find $2\alpha^2 = 1$, so $\alpha = 1/\sqrt{2}$, and we have

$$\pm \frac{1}{2} \hbar \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}.$$

(b) [10] If you measured $S_x$, what is the probability that you get each of the possible eigenvalues you found in part (a)? What would be the state vector afterwards?

The probabilities are given by

$$P(\pm \frac{1}{2} \hbar) = \langle \pm \frac{1}{2} \hbar | \Psi \rangle^2 = \frac{1}{2} \left| 1 - 1 \left( \frac{1}{2} + \frac{2}{3} i \right) \right|^2 = \frac{1}{2} \left[ 1 + \left( \frac{1}{2} \right)^2 \right] = \frac{13}{18},$$

$$P(-\frac{1}{2} \hbar) = \langle -\frac{1}{2} \hbar | \Psi \rangle^2 = \frac{1}{2} \left| 1 - 1 \left( \frac{1}{2} + \frac{2}{3} i \right) \right|^2 = \frac{1}{2} \left[ 1 + \left( \frac{1}{2} \right)^2 \right] = \frac{13}{18}.$$

Since there are no states with degenerate eigenvalues, you must (up to a phase) end up in these eigenstates, so in the first case you will end up in the state $\left| \pm \frac{1}{2} \hbar \right\rangle$ and in the latter case $\left| -\frac{1}{2} \hbar \right\rangle$. 
### Possibly Helpful Formulas:

<table>
<thead>
<tr>
<th>Harmonic Oscillator</th>
<th>Radial Wave Functions</th>
<th>Spherical Harmonics</th>
<th>Hydrogen Energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger)$</td>
<td>$R_{10}(r) = \frac{2e^{-r/\alpha}}{\sqrt{\alpha^2}}$</td>
<td>$Y_0^0(\theta, \phi) = \frac{\sqrt{3}}{2\sqrt{\pi}} \cos \theta$</td>
<td>$E = -\frac{13.6 \text{ eV}}{n^2}$</td>
</tr>
<tr>
<td>$P = i\sqrt{\frac{\hbar}{2m\omega}} (a^\dagger - a)$</td>
<td>$R_{20}(r) = \frac{e^{-r/\alpha}}{\sqrt{2\alpha}} \left(1 - \frac{r}{2\alpha}\right)$</td>
<td>$Y_2^0(\theta, \phi) = \frac{\sqrt{5}}{4\sqrt{\pi}} (3\cos^2 \theta - 1)$</td>
<td></td>
</tr>
<tr>
<td>$a</td>
<td>n \rangle = \sqrt{n}</td>
<td>n - 1 \rangle$</td>
<td>$R_{21}(r) = \frac{re^{-r/\alpha}}{2\sqrt{6\alpha^2}}$</td>
</tr>
<tr>
<td>$a^\dagger</td>
<td>n \rangle = \sqrt{n+1}</td>
<td>n + 1 \rangle$</td>
<td></td>
</tr>
</tbody>
</table>

### Possibly Helpful Integrals:

Definite Integrals: $n$ and $p$ are assumed to be positive integers

\[
\int_0^\infty x^n e^{-ax} \, dx = \frac{n!}{a^{n+1}},
\]

\[
\int_0^a \sin \left(\frac{\pi nx}{a}\right) dx = \begin{cases} \frac{2a}{\pi} & \text{if } n \text{ odd}, \\ 0 & \text{if } n \text{ even}. \end{cases}
\]

\[
\int_0^a \sin^2 \left(\frac{\pi nx}{a}\right) dx = \begin{cases} \frac{4p^2_a}{\pi(n^4 p^4 - n^2)} & \text{if } n \text{ odd}, \\ 0 & \text{if } n \text{ even}. \end{cases}
\]

\[
\int_0^a \sin^2 \left(\frac{\pi nx}{a}\right) \sin^2 \left(\frac{\pi px}{a}\right) dx = a \left(\frac{1}{4} + \frac{1}{8} \delta_{np}\right).
\]

### Possibly helpful sums:

\[
\sum_{n=1}^{\infty} \frac{1}{n^k} = \zeta(k), \quad \sum_{n=1}^{\infty} \frac{(-1)^k}{n^k} = (2^{1-k} - 1)\zeta(k), \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}.
\]