1. [15] A particle of mass $m$ and initial momentum $\hbar k$ moving in the $+z$ direction scatters from a weak potential $V(\mathbf{r}) = \lambda \delta(z) \exp \left[ -\frac{1}{\alpha} \left( x^2 + y^2 \right) \right]$. Calculate the differential cross-section $d\sigma/d\Omega$ and total cross-section $\sigma$ in the first Born approximation. Note that the potential is not spherically symmetric, and hence you cannot choose the momentum change $K$ to be along the $z$-axis. I recommend working in Cartesian coordinates. For $\sigma$, you may leave one integral undone.

2. [20] A particle of mass $m$ lies in the spherical infinite square well, with potential

$$V(\mathbf{r}) = \begin{cases} 0 & \text{if } |\mathbf{r}| < R, \\ \infty & \text{if } |\mathbf{r}| > R. \end{cases}$$

The radius $R$ of the confining sphere will actually be a function of time, so that it changes.

(a) [10] The ground state is given in the allowed region $r < R$ by

$$\psi(r) = \frac{N}{r} \sin \left( \frac{\pi r}{R} \right)$$

Show that this is, in fact, an eigenstate of the Hamiltonian, and determine the energy and the correct normalization factor $N$.

(b) [10] The radius $R$ of the allowed region is slowly decreased from $R$ to $\frac{1}{2} R$. What is the probability that it will remain in the ground state? The radius is then suddenly increased from $\frac{1}{2} R$ to $R$. What is the probability now that it remains in the ground state?
3. [15] A particle is in a one-dimensional infinite square well, with allowed region \(0 < x < a\). It is initially in the ground, or \(|n = 1\rangle\) state. At \(t = 0\), a perturbation of the form \(W = A Pe^{-\lambda t}\), where \(A\) is a small positive number and \(P\) is the momentum operator, is turned on. Calculate to leading order (that's order \(A^2\)), the probability that the particle ends up in the state \(|n\rangle\) at \(t = \infty\) for all \(n > 1\).

4. [15] A particle of mass \(m\) in one dimension is in the bound state of the Dirac delta function. This state has wave function and energy given by

\[
\psi_g(x) = \sqrt{\alpha} e^{-\alpha|x|}, \quad E_g = -\varepsilon,
\]

The values of \(\alpha\) and \(\varepsilon\) depend on the mass and the strength of the delta function. It is then affected by a time-dependent perturbation of the form \(W(t) = \beta P \cos(\omega t)\), where \(P\) is the momentum operator and \(\beta\) is very small. The frequency is high enough such that \(\hbar \omega \gg \varepsilon\).

(a) [2] Write \(W(t)\) in the form \(W(t) = W e^{-i\omega t} + W^* e^{i\omega t}\). What is \(W\)?

(b) [5] For sufficiently high energies, the potential is negligible, and the eigenstates are approximately momentum eigenstates \(|k\rangle\). Work out the matrix element \(\langle k | W | \psi_g \rangle\) for these high energy final states. Hint – let any momentum operators act to the left.

(c) [8] Calculate the rate \(\Gamma \left( \left| \psi_g \right\rangle \rightarrow \left| k \right\rangle \right)\) for a given final state wave number \(k\).

Integrate it over \(k\) to get the rate for the transition.

5. [20] A system consists of a superposition of a zero photon state and two one photon states, so that

\[
\left| \Psi(t = 0) \right\rangle = N \left( \sqrt{2} \left| 0 \right\rangle - \left| q, 1 \right\rangle - i \left| q, 2 \right\rangle \right)
\]

where \(N\) is a normalization constant, \(q = q\hat{z}\), and the two polarizations are \(\varepsilon(q, 1) = \hat{x}\) and \(\varepsilon(q, 2) = \hat{y}\).

(a) [3] What is the normalization constant \(N\)?

(b) [5] What is the state vector \(\left| \Psi(t) \right\rangle\) at an arbitrary time? Work in the convention where the energy of the vacuum is 0.

(c) [12] what is the expectation value of the electric field \(\left\langle E(r) \right\rangle\) at arbitrary time and arbitrary position for the state you found in part (b)?
6. [20] An electron is trapped in a 3D infinite square well, with allowed region $0 < x < a, 0 < y < a, 0 < z < a$. It is in the quantum state $|n_x, n_y, n_z\rangle = |1,1,4\rangle$.

(a) [12] Calculate every non-vanishing matrix element of the form $\langle n_x, n_y, n_z | R | 1,1,4 \rangle$ where the final state is lower in energy than the initial state (there will be very few of these).

(b) [8] Calculate the decay rate $\Gamma(\{1,1,4\} \rightarrow |n_x, n_y, n_z\rangle)$ for this decay in the dipole approximation for every possible final state. What is the ratio of the two rates?

7. [30] A hydrogen atom is at rest in a strong magnetic field oriented in the z-direction. This field adds an additional perturbation to the hydrogen atom as given in Eq. (9.23):

$$ W = \frac{eB}{m} \left( S_z + \frac{1}{2} L_z \right) $$

The field is sufficiently strong that we can neglect any hyperfine effects. The atom is initially in the state $|n, l, m, m_\ell\rangle = |1,0,0,\frac{1}{2}\rangle$, but it will transition to $|1,0,0,-\frac{1}{2}\rangle$.

(a) [2] Explain why this transition cannot occur in the electric dipole approximation.

(b) [8] Find all non-vanishing matrix elements coming from the magnetic dipole transition $\langle F | (S + \frac{1}{2} L) | I \rangle$.

(c) [20] Find a formula for the differential decay rate $d\Gamma/d\Omega$ for this decay into a particular polarization $\varepsilon$. You do not need to sum it or integrate it over polarizations.

8. [15] The Dirac equation in the presence of electromagnetic fields is given by

$$ i\hbar \frac{\partial}{\partial t} \Psi = \left\{ c\alpha \cdot \left[ P + eA(R,t) \right] - eU(R,t) + mc^2 \beta \right\} \Psi $$

where $P = -i\hbar \nabla$. Prove or disprove: The Dirac equation is invariant under gauge transformations.
Definite Integrals:

For each of the following formulas, $m$ and $n$ and $p$ are assumed to be positive integers.

\[ \int_0^\infty e^{-Ax}x^n\,dx = \frac{\Gamma(n+1)}{A^{n+1}} \quad \text{if } \Re(A) > 0, \]

\[ \int_0^\infty e^{-Ax^2}x^n\,dx = \frac{\Gamma(n+1/2)}{2A^{n+1/2}} \quad \text{if } \Re(A) > 0, \]

\[ \Gamma(n+1) = n!, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}, \quad \Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi}. \]

\[ \int_0^a \sin\left(\frac{\pi nx}{a}\right)\,dx = \begin{cases} 2a/n\pi & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even}, \end{cases} \]

\[ \int_0^a \cos\left(\frac{\pi nx}{a}\right)\,dx = 0 \]

\[ \int_0^a \cos\left(\frac{\pi nx}{a}\right)\cos\left(\frac{\pi mx}{a}\right)\,dx = \int_0^a \sin\left(\frac{\pi nx}{a}\right)\sin\left(\frac{\pi mx}{a}\right)\,dx = \frac{1}{2}a\delta_{nm} \]

\[ \int_0^a \sin\left(\frac{\pi nx}{a}\right)\cos\left(\frac{\pi mx}{a}\right)\,dx = \begin{cases} 2na/\pi (n^2 - m^2) & \text{if } n + m \text{ is odd} \\ 0 & \text{if } n + m \text{ is even} \end{cases} \]

\[ \int_0^a \sin\left(\frac{\pi nx}{a}\right)\sin\left(\frac{\pi mx}{a}\right)\,dx = \begin{cases} -4a^2mn/\left[\pi^2(n^2 - m^2)^2\right] & \text{if } n + m \text{ is odd} \\ \frac{1}{4}a^2\delta_{nm} & \text{if } n + m \text{ is even} \end{cases} \]

\[ \int_0^a \cos\left(\frac{\pi nx}{a}\right)\cos\left(\frac{\pi mx}{a}\right)\,dx = \frac{a^2n(-1)^{m+n}}{\pi(m^2 - n^2)} \]

\[ \int_0^a \cos\left(\frac{\pi nx}{a}\right)\cos\left(\frac{\pi mx}{a}\right)\,dx = \begin{cases} -2a^2(m^2 + n^2)/\left[\pi^2(n^2 - m^2)^2\right] & \text{if } n + m \text{ is odd} \\ \frac{1}{4}a^2\delta_{nm} & \text{if } n + m \text{ is even} \end{cases} \]