1. An electron in a magnetic field in the $xz$-plane will have Hamiltonian given by

$$H = \frac{e}{m} \mathbf{B} \cdot \mathbf{S}, \quad \text{where} \quad S_x |\pm\rangle = \frac{1}{2} \hbar |\mp\rangle, \quad S_z |\pm\rangle = \pm \frac{1}{2} \hbar |\pm\rangle.$$ 

At $t=0$, the magnetic field is $\mathbf{B} = B\hat{z}$, and the particle is in the $|\rangle$ state. As time increases, the magnetic field rotates (while staying the same strength) until it is in the direction $\mathbf{B} = \frac{1}{2} B \left( \sqrt{3} \hat{x} - \hat{z} \right)$. Show that $|\phi_1\rangle = \frac{1}{2} |+\rangle + \frac{1}{2} \sqrt{3} |\rangle$ and $|\phi_2\rangle = \frac{1}{2} |\rangle - \frac{1}{2} \sqrt{3} |+\rangle$ are eigenstates of the Hamiltonian, and find the probability it ends in each of these states if the magnetic field rotates (a) suddenly, or (b) slowly?

2. A particle is in the ground state $|0\rangle$ of a 1D harmonic oscillator with frequency $\omega_0$. A small perturbation of the form $W(t) = \alpha X^3 e^{-\alpha t^2}$ is applied from $t = -\infty$ to $t = +\infty$. Calculate to leading order (second order in $\alpha$) the probability that it transitions to any possible state $|n\rangle$ where $n > 0$.

3. A particle of mass $m$ in the spherical infinite square well of radius $a$, if in the $|n,0,0\rangle$ state, has wave function and energy

$$\psi_{n00}(r) = \frac{\sin(\pi nr/a)}{r \sqrt{2\pi a}}, \quad E_{n00} = \frac{\pi^2 n^2 \hbar^2}{2ma^2}.$$ 

A particle is initially in the $|2,0,0\rangle$ and is then subjected to a perturbation of the form $W(t) = \varepsilon \cos(\pi r/a) \cos(\omega t)$. To leading order, to which states $|n,0,0\rangle$ can it transition? Write the transition rate. For which frequencies $\omega$ will each of these transitions occur?

4. Compute the expectation value $\langle 0|\mathbf{E}(0) \cdot \mathbf{E}(r)|0\rangle$ where $|0\rangle$ is the vacuum state with no photons and $r$ is an arbitrary position. Simplify as much as possible. You may leave one one-dimensional integral undone in the end.

5. A particle is in a 3D-symmetric harmonic oscillator with frequency $\omega_0$ in the state $|3,2,0\rangle$. In the dipole approximation, which states can it decay to? Find the decay rate to each of these states, and the branching ratio.
### Possibly Helpful Formulas:

<table>
<thead>
<tr>
<th>Formula</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1D H.O.:</strong> $V(x) = \frac{1}{2} m \omega_0^2 x^2$</td>
<td>Harmonic Perturbations: $W(t) = W e^{-i\omega t} + W^* e^{i\omega t}$, $W_{\phi_1} \equiv \langle \phi_1</td>
</tr>
<tr>
<td>$X = \sqrt{\frac{\hbar}{2 m \omega_0}} (a + a^\dagger)$</td>
<td>$\Gamma(I \rightarrow F) = \begin{cases} 2\pi h^{-1} \left</td>
</tr>
<tr>
<td>$a</td>
<td>n\rangle = \sqrt{n}</td>
</tr>
<tr>
<td>$a^\dagger</td>
<td>n\rangle = \sqrt{n+1}</td>
</tr>
<tr>
<td>$E(r) = \sum_{k,\sigma} \sqrt{\frac{\hbar \omega}{2 e \nu}} e^{-i(\epsilon_{k,\sigma} e^{i\mathbf{k} \cdot \mathbf{r}} - \epsilon_{k,\sigma}^* e^{-i\mathbf{k} \cdot \mathbf{r}})}$</td>
<td>Electric field operator</td>
</tr>
<tr>
<td>$\phi_n(x) = \sqrt{\frac{2}{a}} \sin \left( \frac{\pi n x}{a} \right)$</td>
<td>Spont. Decay $\Gamma = \frac{4\alpha}{3 c^2} \omega_{\nu e}^3 \left</td>
</tr>
</tbody>
</table>

### Possibly Helpful Integrals:

The formulas below assume $n$, $p$, and $q$ are positive integers

\[
\int_{-\infty}^{\infty} e^{-Ax^2 + Bx} \, dx = \frac{\sqrt{\pi}}{A} e^{B^2/4A}, \quad \int_0^\infty x^a e^{-ax} \, dx = n! a^{-(n+1)}.
\]

\[
\int_0^a \sin(\pi nx/a) \sin(\pi px/a) \, dx = \int_0^a \cos(\pi nx/a) \cos(\pi px/a) \, dx = \frac{1}{2} a \delta_{np} ,
\]

\[
\int_0^a \sin(\pi nx/a) \sin(\pi px/a) \cos(\pi qx/a) \, dx = \frac{1}{4} a \left( \delta_{n+q,p} + \delta_{p+q,n} - \delta_{n+q,p} \right),
\]

\[
\int_0^a \cos(\pi nx/a) \cos(\pi px/a) \cos(\pi qx/a) \, dx = \frac{1}{4} a \left( \delta_{n+q,p} + \delta_{p+q,n} + \delta_{n+q,p} \right).
\]