1. [15] A particle of mass \( m \) lies in a potential \( V(x) = \begin{cases} -\alpha x & x < 0, \\ \beta x & x > 0. \end{cases} \)

Using the WKB method, estimate the energy of the \( n \)’th bound quantum state.

2. [20] Estimate the ground-state energy of a particle of mass \( m \) in the potential

\[
V(x) = \begin{cases} \frac{Bx}{x > 0}, \\ \infty & x < 0, \end{cases}
\]

using the variational principle with trial wave function \( \psi(x) = xe^{-Ax} \) in the allowed region.

3. [25] A particle is in the Hamiltonian

\[
H_0 = \frac{1}{2m} \left( P_x^2 + P_y^2 \right) + \frac{1}{2} m \omega^2 \left( X^2 + Y^2 \right) + bXY,
\]

where \( b \) is small.

(a) [2] What are the eigenstates and corresponding energies if \( b = 0 \)?
(b) [10] Find the ground state energy to second order in \( b \) and the eigenstate to first order in \( b \).
(c) [13] For the first excited states, find the eigenstates to leading (zeroth) order in \( b \) and the corresponding energy to first order in \( b \).

4. [20] An electron is trapped in a Coulomb potential of the form \( V_C(r) = A/r \). It is in one of the \( l = 1 \) states, so its space wave function looks like \( \psi(r) = R(r)Y_1^m(\theta, \phi) \) where \( Y_1^m(\theta, \phi) \) is a spherical harmonic and \( R(r) \) is a radial wave function.

(a) [10] Given that \( l = 1 \), what are the possible values of \( j \), the total angular momentum quantum number? For each of these values of \( j \), work out the corresponding eigenvalue of \( \mathbf{L} \cdot \mathbf{S} \).
(b) [10] Find the energy splitting \( \Delta \epsilon' \) between the different states you found in part (a) due to spin-orbit coupling. Since you don’t know the radial wave function, you will have to leave one integral undone.

5. [20] A particle of mass \( \mu \) with momentum \( \hbar k \) scatters from a potential of the form

\[
V(r) = A\delta(x)\delta(y)e^{-|z|}.
\]

Using the first Born approximation, find the differential and total cross-section.
### Possibly Helpful Formulas:

<table>
<thead>
<tr>
<th>Spin-Orbit Coupling</th>
<th>Born Approximation</th>
<th>1D Harmonic Oscillator</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W_{SO} = \frac{g}{4m^2c^2} \frac{1}{r} \frac{dV_c(r)}{dr} \mathbf{L} \cdot \mathbf{S} )</td>
<td>( \frac{d\sigma}{d\Omega} = \frac{\mu^2}{4\pi^2\hbar^2} \left</td>
<td>\int d^3r V(r) e^{-\mathbf{k} \cdot \mathbf{r}} \right</td>
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</tbody>
</table>

#### Definite integral:

\[ \int_0^\infty x^n e^{-ax} \, dx = \alpha^{-n} n! \]

\[ \int dx (a + bx) = ax + \frac{1}{2} bx^2 + C, \quad \int dx \sqrt{a + bx} = \frac{2}{3b} (a + bx)^{3/2} + C, \]

#### Indefinite Integrals:

\[ \int \frac{dx}{a + bx} = \frac{1}{b} \ln(a + bx) + C, \quad \int \frac{dx}{\sqrt{a + bx}} = \frac{2}{b} \sqrt{a + bx} + C. \]

\[ \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C, \quad \int \frac{dx}{(x^2 + a^2)^2} = \frac{1}{2a^3} \left[ \tan^{-1}\left(\frac{x}{a}\right) + \frac{ax}{a^2 + x^2} \right] \]