Lecture 1: Functions

Definition: A function $f$ is a rule that assigns to each $x$ in a set $D$ exactly one element, called $f(x)$, in a set $E$.

* $D$ is the domain.
* $E$ is the range.
* $x$ is the independent variable.
* $f$ is the dependent variable.

Example (formula):
Volume of a sphere of radius $r$:
$$V(r) = \frac{4}{3} \pi r^3.$$

Domain: $r \geq 0$, $E: r \geq 0$
Range: $V \geq 0$, $E: V \geq 0$.

Set builder notation: $\{ r \in \mathbb{R} : r \geq 0 \}$

The graph of a function is the set of ordered pairs:
$$\{ (r, V(r)) : r \in D \}.$$

Example (table):

<table>
<thead>
<tr>
<th>School Year</th>
<th># Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1285</td>
</tr>
<tr>
<td>2</td>
<td>1127</td>
</tr>
<tr>
<td>3</td>
<td>1118</td>
</tr>
<tr>
<td>4</td>
<td>1113</td>
</tr>
</tbody>
</table>

$D = \{1, 2, 3, 4, 5\}$
$E = \{1285, 1127, 1118, 1113\}$
Example:
Plot of distance \( d(t) \) travelled by a car:

\[
d(t) = \begin{cases} 
60t & 0 \leq t \leq 1 \\
60 & 1 < t < 3 \\
30(t - 3) & 3 \leq t \leq 5
\end{cases}
\]

\[
\text{Domain:} \quad \mathbb{R}_+ : 0 \leq t \leq 5 \\
\text{Range:} \quad \mathbb{Z}_+ : 0 \leq d \leq 90.
\]

Sketch a graph of the velocity \( v(t) \):

\[
v(t) = \begin{cases} 
60 & 0 \leq t \leq 1 \\
0 & 1 < t < 3 \\
30 & 3 \leq t \leq 5
\end{cases}
\]

Definition: A function \( f \) is increasing on an interval \( I \) if
\[
f(x_1) < f(x_2) \quad \text{when} \quad x_1 < x_2 \quad \text{in} \quad I.
\]
It is called decreasing on \( I \) if
\[
f(x_1) > f(x_2) \quad \text{when} \quad x_1 < x_2 \quad \text{in} \quad I.
\]

In the example above, \( d(t) \) is increasing on \([0, 1]\) and \([3, 5]\).
example:

If \( f(x) = ax^2 + bx + c \), what is

\[
\begin{align*}
\frac{f(x+h) - f(x)}{h} &= \frac{a(x+h)^2 + b(x+h) + c - (ax^2 + bx + c)}{h} \\
&= \frac{ax^2 + 2axh + ah^2 + bx + bh - ax^2 - bx - c}{h} \\
&= \frac{2axh + ah^2 + bh}{h} \\
&= \frac{h(2ax + ah + b)}{h} \\
&= 2ax + ah + b
\end{align*}
\]

example:

Suppose water fills a barrel at a constant rate:

Sketch a graph of the height of the water as a function of time.
**Symmetry:**

A function is odd if $f(x) = -f(-x)$.

A function is even if $f(x) = f(-x)$.

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**Example:**

Which functions are odd or even?

- $f(x) = x^2 + 2$ (even)
- $f(x) = 4x^3 + 3x$ (odd)
- $f(x) = x^3 + 1$ (odd)