Lecture 4: Tangents and Limits

Tangent line: The tangent line passing through a point \((x_0, f(x_0))\) lying on the graph of a function \(f(x)\) "touche"s \((x_0, f(x_0))\).

Example:
Find an equation of the tangent line to the function \(f(x) = x^3\) at \((1, 1)\).

Solution:
The idea is to generate a sequence of approximations that get better and better.

\[
\begin{align*}
\frac{f(1 + h) - f(1)}{h} & = \text{rise} = \frac{8 - 1}{1} = 7 \\
\frac{f(1 + \frac{1}{2}) - f(1)}{\frac{1}{2}} & = \text{rise} = \frac{\frac{27}{8} - 1}{\frac{1}{2}} = 4.75 \\
\frac{f(1 + \frac{1}{4}) - f(1)}{\frac{1}{4}} & = \text{rise} = \frac{\frac{125}{16} - 1}{\frac{1}{4}} = 61 = 3.81 \\
\frac{f(1 + \frac{1}{16}) - f(1)}{\frac{1}{16}} & = \text{rise} = \frac{1}{1} = 1
\end{align*}
\]
Let's make a new function which gives slope of secant lines:

\[ g(h) = \frac{f(1+h) - f(1)}{h} = \frac{\text{rise}}{\text{run}} \]

\[ = \frac{(1+h)^3 - 1}{h} \]

\[ = \frac{x^3 + 3hx^2 + 3h^2x - x^3}{h} \]

\[ = 3 + h + h^2 \]

As \( h \) goes to zero it follows that:

\[ \lim_{h \to 0} g(h) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = 3 = \text{slope at tangent line.} \]

However, \( g(0) = \frac{f(1+0) - f(1)}{0} = \frac{0}{0} \) (Undefined).

\[ \Rightarrow y - 1 = 3(x-1) \]

\[ \Rightarrow y = 3(x-1) + 1 \text{ Equation of tangent line.} \]
**Limit** Suppose \( f(x) \) is defined when \( x \) is near a number \( a \). Then we write
\[
\lim_{{x \to a}} f(x) = L
\]
if \( f(x) \) gets arbitrarily close to \( L \) by restricting \( x \) to be sufficiently close to \( a \).

**Example:**
1. \( \lim_{{x \to 2}} 3x^2 = 3 \cdot 8 = 24 \)
2. \( \lim_{{x \to 3}} \frac{x^2 - 9}{x - 3} \), "dumb" thing \( \frac{3^2 - 9}{3 - 3} = \frac{0}{0} \rightarrow \text{Undefined} \)
   
   \[
   \lim_{{x \to 3}} (x-3)(x+3) = \lim_{{x \to 3}} x + 3 = 6.
   \]
3. \( \lim_{{x \to 0}} \frac{\sqrt{x^2 + 9} - 3}{x^2} \), "dumb" thing \( \frac{\sqrt{9} - 3}{0} = \frac{0}{0} \rightarrow \text{Undefined}! \)
   
   \[
   \lim_{{x \to 0}} \frac{\sqrt{x^2 + 9} - 3}{x^2} = \lim_{{x \to 0}} \frac{x^2 + 9 - 9}{x^2 (\sqrt{x^2 + 9} + 3)} = \lim_{{x \to 0}} \frac{1}{\sqrt{x^2 + 9} + 3} = \frac{1}{6}.
   \]
Velocity: 

The distance of an object in free fall is given by:

\[ d(t) = 4.9 t^2 \] (meters).

How fast is the object falling after 2 seconds?

\[ \text{rate} \times \text{time} = \text{distance} \]

\[ \Rightarrow \text{rate} = \frac{\text{distance}}{\text{time}} \]

<table>
<thead>
<tr>
<th>Interval of time</th>
<th>distance</th>
<th>Average Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2 \leq t \leq 3)</td>
<td>(d(3) - d(2) = 21.5)</td>
<td>(2.5/1 = 24.5 \text{ m/s})</td>
</tr>
<tr>
<td>(2 \leq t \leq 2.1)</td>
<td>2.009</td>
<td>20.09 m/s</td>
</tr>
<tr>
<td>(2 \leq t \leq 2.01)</td>
<td>0.196</td>
<td>19.649</td>
</tr>
<tr>
<td>(2 \leq t \leq 2.001)</td>
<td>0.0196</td>
<td>19.6609</td>
</tr>
</tbody>
</table>

Instantaneous Velocity \(\approx 19.6 \text{ m/s}\).

The exact result is:

\[ v(2) = \lim_{\Delta t \to 0} \frac{d(2 + \Delta t) - d(2)}{\Delta t} \]

\[ = \lim_{\Delta t \to 0} \frac{4.9(2 + \Delta t)^2 - 4.9 \cdot 2^2}{\Delta t} \]

\[ = \lim_{\Delta t \to 0} \frac{4.9(4 + 2\Delta t + \Delta t^2) - 4.9 \cdot 4}{\Delta t} \]

\[ = \lim_{\Delta t \to 0} \frac{9.8 \cdot 2 + 4.9 \Delta}{\Delta} = 19.6 \text{ m/s}. \]
Harder Limits:

1. What is \( \lim_{{x \to 0}} \frac{1 - \cos(x)}{x} ? \)

"Dunh" thing: \( 1 - \cos(0) = 0 \), undefined, do more work!!

\[
\begin{array}{c|c}
 x & \frac{1 - \cos(x)}{x} \\
-0.005 & -0.1 \\
0.005 & 0.1 \\
-0.0499583 & -1 \\
0.0499583 & 1 \\
-0.45968 & -1 \\
0.45968 & 1 \\
\end{array}
\]

\[
\lim_{{x \to 0}} \frac{1 - \cos(x)}{x} = 0
\]

\[
\lim_{{x \to \pi}} \frac{1 - \cos(x)}{x} = \frac{2}{\pi}
\]

\[
\lim_{{x \to 2\pi}} \frac{1 - \cos(x)}{x} = 0
\]

2. \( \lim_{{x \to 0}} \sin \left( \frac{1}{x} \right) = \text{Does not exist.} \)

\[
\Rightarrow \text{Infinite \# \ of \ zeros \ in \ } [-1, 1] \]
One Sided Limits: We write
\[ \lim_{x \to a^-} f(x) = L \quad \text{or} \quad \lim_{x \to a^+} f(x) = L \]
and say the **left-hand limit** (or right-hand limit) as \( x \) approaches \( a \) is equal to \( L \) if \( f(x) \)
gets arbitrarily close to \( L \) by taking \( x \) sufficiently close to \( a \) with \( x \) less (greater) than \( a \).

**Example:**

\[ f(x) \]

\[ \begin{array}{c}
\text{Graph showing limit values.}
\end{array} \]

\[ \lim_{x \to -2^-} f(x) = 3, \quad \lim_{x \to 2^-} f(x) \text{ does not exist} \]

\[ \lim_{x \to 1} f(x) = 2, \quad \lim_{x \to 2^+} f(x) = \frac{3}{2} \]

\[ \lim_{x \to 2^+} f(x) = 4 \]

**Theorem:**
\[ \lim_{x \to a^-} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^+} f(x) = L \]
Infinite Limits:

\[ \lim_{x \to a} f(x) = \infty \quad \text{or} \quad \lim_{x \to a} f(x) = -\infty \]

means the values of \( f(x) \) can be made arbitrarily large (or arbitrarily large negative) by taking \( x \) sufficiently close to \( a \).