Lecture 5: Limit Laws and Continuity

**Laws:**

Suppose $c$ is a constant and 

\[ \lim_{x \to a} f(x) \] and \[ \lim_{x \to a} g(x) \]

exist. Then,

1. \[ \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x). \]

2. \[ \lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x). \]

3. \[ \lim_{x \to a} [c \cdot f(x)] = c \cdot \lim_{x \to a} f(x). \]

4. \[ \lim_{x \to a} [f(x) \cdot g(x)] = f(a) \cdot g(a). \]

5. \[ \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \quad \text{if} \quad \lim_{x \to a} g(x) \neq 0. \]

**Example:**
1. \[ \lim_{x \to -2} \left( f(x) + 5g(x) \right) = \lim_{x \to -2} f(x) + 5 \lim_{x \to -2} g(x) \]
   
   \[ = \lim_{x \to -2} f(x) + 5 \cdot 1 \]
   
   \[ = \lim_{x \to -2} f(x) + 5 \cdot 1 \]
   
   \[ = 7 \]

2. \[ \lim_{x \to 1} f(x) \cdot g(x) \]
   
   \[ = \lim_{x \to 1} f(x) \cdot \lim_{x \to 1} g(x) \]
   
   \[ = -1 \cdot 2 = -2 \]

   \[ = \lim_{x \to 1} f(x) \cdot \lim_{x \to 1} g(x) \]
   
   \[ = -1 \cdot 3 = 3 \]

   \[ \Rightarrow \lim \text{ does not exist.} \]

More Laws:

4. \[ \lim_{x \to a} f(x)^n = \left[ \lim_{x \to a} f(x) \right]^n \quad n \text{ is an integer.} \]

Proof:

\[ \lim_{x \to a} f(x)^n = \lim_{x \to a} f(x) \cdot \lim_{x \to a} f(x)^{n-1} \]

\[ = \lim_{x \to a} f(x) \cdot \lim_{x \to a} f(x)^{n-1} \]

\[ = \left[ \lim_{x \to a} f(x) \right] \cdot \lim_{x \to a} f(x)^{n-2} \]

\[ = \left[ \lim_{x \to a} f(x) \right]^n \]
2. \( \lim_{x \to a} c = c \)

3. \( \lim_{x \to a} x = a \)

4. \( \lim_{x \to a} x^n = a^n \).

   \text{Proof:} \\
   \lim_{x \to a} x^n = \left[ \lim_{x \to a} x \right]^n = a^n.

5. \( \lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} \).

\text{Theorem} \quad \lim_{x \to a} f(x) = L \iff \lim_{x \to a^{-}} f(x) = L = \lim_{x \to a^{+}} f(x) \quad (x \to a)

\text{Example:}

1. \( \lim_{x \to 0} |x| = \lim_{x \to 0^-} x = \lim_{x \to 0^+} (-x) = 0. \)

2. \( \lim_{x \to 0} \frac{x}{|x|} ? \)

\( \lim_{x \to 0^{+}} \frac{x}{|x|} = 1 \)
\( \lim_{x \to 0^{-}} \frac{x}{|x|} = -1 \)

\( \text{The limit does not exist.} \)
3. Find \( k \) so that the following limit exists

\[
\lim_{x \to 4} f(x) = 0
\]

if

\[
f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ k - 2x & \text{if } x \leq 4 \end{cases}
\]

\[
\lim_{x \to 4^+} f(x) = 0
\]

\[
\lim_{x \to 4^-} f(x) = k - 8
\]

Therefore, if \( k = 8 \) it follows that

\[
\lim_{x \to 4^+} f(x) = 1, \quad \lim_{x \to 4^-} f(x) = 0
\]

\[\Rightarrow \lim_{x \to 4} f(x) = 0.\]

**Theorem:** If \( f(x) \leq g(x) \leq h(x) \) and

\[
\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L
\]

then

\[
\lim_{x \to a} g(x) = L.
\]

**Example:**

\[
\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)
\]

\[-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2
\]

\[\Rightarrow \lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.
\]
Definition: A function $f$ is continuous at $a$ if
\[ \lim_{x \to a} f(x) = f(a). \]

* This definition assumes:
  1. $f(a)$ is defined
  2. $\lim_{x \to a} f(x)$ exists.

\[ f(x) = \begin{cases} 
  cx^2 + 2x, & \text{if } x < 2 \\
  x^3 - cx^2 & \text{if } x \geq 2
\end{cases} \]

$x = -3$: Discontinuity, function is not defined
$x = 1$: Limit does not exist, discontinuity
$x = 3$: Limit does not match function value, discontinuity
$x = 5$: Function is not defined

Definition: A function $f$ is continuous on an interval if it is continuous at every number in the interval.

Example:
Find a value of $k$ which makes the following function continuous:
\[ f(x) = \begin{cases} 
  cx^2 + 2x, & \text{if } x < 2 \\
  x^3 - cx^2 & \text{if } x \geq 2
\end{cases} \]
\[ \lim_{x \to 2^+} f(x) = 8 - 2c \]
\[ \lim_{x \to 2^-} f(x) = 4c^2 + 4 \]

If this function is continuous then:

\[ 8 - 2c = 4c^2 + 4 \]

\[ \implies 2c^2 + c - 2 = 0 \]
\[ \implies c = \frac{-1 \pm \sqrt{1 + 16}}{4} \]
\[ c = -1 + \frac{1}{4}, \quad -1 - \frac{1}{4} \]

Theorem: The following functions are continuous on their domains:
- polynomials
- rational functions
- power functions
- trigonometric functions

Theorem: If \( f \) is continuous at \( b \) and \( \lim_{x \to a} g(x) = b \), then

\[ \lim_{x \to a} f(g(x)) = f(b) \]

i.e.,

\[ \lim_{x \to a} f(g(x)) = f \left( \lim_{x \to a} g(x) \right) \]
Theorem: If \( \lim_{x \to a} f(x) = b \) then \( \lim_{x \to a} [f(x)]^n = b^n \).

**Proof:** Let \( g(x) = x^n \). Then \( g \) is continuous and

\[
\lim_{x \to a} [f(x)]^n = \lim_{x \to a} g(f(x)) = g(\lim_{x \to a} f(x)) = g(b) = b^n.
\]

Theorem: If \( g \) is continuous at \( a \) and \( f \) is continuous at \( g(a) \), then the composite function \( h(x) = f(g(x)) \) is continuous at \( a \).

**Proof:**

\[
\lim_{x \to a} h(x) = \lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(g(a)) = h(a).
\]

**Theorem (The Intermediate Value Theorem):** Suppose that \( f \) is continuous on the interval \([a, b]\) and let \( N \) be any number between \( f(a) \) and \( f(b) \) where \( f(a) \neq f(b) \). Then there exists \( c \) in \([a, b]\) such that \( f(c) = N \).
Example:

Show that there is a solution to the following equation $2^x = x^2$.

**Proof:**

Let $g(x) = 2^x - x^2$.

$g(1) = 2 - 1 = 1$

$g(-1) = 1/2 - 1 = -1/2$

By the intermediate value theorem there exists $c$ in $[-1, 1]$ such that $g(c) = 0$. 