Please solve 3 of the following 4 problems. If you solve all 4, then you must indicate which 3 you want graded.

1. A pearl of mass $m$ with a hole in it can slide, under the influence of gravity, along a frictionless straight wire, which makes an angle $\beta$ ($0 < \beta < \pi/2$) with the vertical (+$z$-axis) and rotates with fixed angular velocity $\omega$.

   (a) Find the constraints of the pearl.
   (b) Find the Lagrange function.
   (c) Find the equation of motion.
   (d) There is a distance $d > 0$, measured from the apex, at which the pearl is in equilibrium. Find $d$. What kind of equilibrium is it?
   (e) Solve the equation of motion under the condition that at $t = 0$ the pearl rests at a distance $d$ from the apex

2. It is sometimes possible to incorporate frictional effects into a Lagrangian. Consider the following Lagrangian:

   $$L = \frac{e^{\gamma t}}{2} \left[ m\dot{q}^2 - kq^2 \right]$$

   where $k$ and $\gamma$ are positive, real constants.

   (a) What is the equation of motion? What kind of system does this Lagrangian describe? Are there any constants of the motion?
   (b) What is the Hamiltonian? Is it a constant of the motion? Is it the total energy?
   (c) Now consider a transformation to a new generalized coordinate $s$ defined by

   $$s = e^{\frac{\gamma}{2} t} q.$$

   This is sometimes called a “point transformation”. Construct the Lagrangian in terms of $s$ and $\dot{s}$, and then answer parts (a) and (b) for the new Lagrangian.
   (d) Solve the equation of motion for $s$. The solution will depend on the value of $\gamma$. List all solutions and the value or range of values for $\gamma$ in which each solution holds.
3. Two particles move in one dimension at the junction of three springs, as shown in the figure. The springs all have unstretched lengths equal to $a$, and the force constants and masses are shown.

Find the eigenfrequencies and normal modes of the system.

4. A hoop of mass $m$ and radius $R$ rolls without slipping down an inclined plane of mass $M$. The inclined plane slides without friction along a horizontal surface. The inclined plane has a fixed angle $\alpha$ with the horizontal surface as shown in the diagram.

(a) Write down the equations of motion for both the hoop and the inclined plane.
(b) Solve the equations, assuming that at $t = 0$, both the hoop and the inclined plane are at rest.