Quantum Mechanics
Graduate Exam

Each of the following questions counts equally. Some useful integrals are given at the end of the exam. Each question is worth 25 points. The points for individual parts are marked in square brackets. Do any four (4) of the following five (5) problems:

1. Imagine a system in which there are just two linearly independent states: 

\[ |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

Suppose the Hamiltonian, represented as a matrix in this basis, is

\[ \hat{H} = \begin{pmatrix} f & g \\ g & f \end{pmatrix}, \]

where \( g \) and \( f \) are real constants.

(a) [9] Find the eigenvalues and (normalized) eigenvectors of this Hamiltonian.

(b) [9] Suppose the system starts out at \( t = 0 \) in state \( |1\rangle \). What is the state at time \( t \)?

(c) [7] What is the probability that it will still be in state \( |1\rangle \) at time \( t = \pi \hbar / 2g \)?

2. A particle of mass \( m \) lies in an asymmetric 2D harmonic oscillator with Hamiltonian

\[ H = \frac{P_x^2 + P_y^2}{2m} + \frac{1}{2} m\omega_0^2 X^2 + 2 m\omega_0^2 Y^2 \]

In addition, there is a small additional perturbation

\[ W = 4\gamma Y P_x^2 \]

(a) [6] Find the eigenstates and energies of the unperturbed Hamiltonian. You do not need to give explicit forms for these eigenstates, it is sufficient to simply label them as, say, \( |pq\rangle \). Check that the ground state and first excited state are non-degenerate, but the second excited states are degenerate.

(b) [10] Find the ground state energy \( E_g \) to second order in \( \gamma \), and the ground-state state vector \( |\psi_g\rangle \) to first order in \( \gamma \).

(c) [9] Find the energies of the second excited states to first order in \( \gamma \), and the corresponding eigenstates to leading order.

Note: For the 1d harmonic oscillator with mass \( m \) and angular frequency \( \omega \), we have

\[ X = \sqrt{\hbar / 2m\omega} \left( a + a^\dagger \right) \quad \text{and} \quad P = i\sqrt{\hbar \omega / 2} \left( a^\dagger - a \right). \]
3. An electron is in the spin state

$$|x\rangle = A\left(\frac{3i}{4}\right)$$

in the standard basis of eigenstates of $S_z$.

(a) [4] Determine the normalization constant $A$.

(b) [8] Find the expectation value of $S_x$, $S_y$, and $S_z$.

(c) [7] Find the uncertainties for these measurements.

(d) [6] Confirm your results are consistent with all three generalized uncertainty relations for these observables.

Possibly Helpful Formulas:

$$S_i = \frac{1}{2} \hbar \sigma_i, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_i^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

4. Two distinguishable particles in one dimension have wave function

$$\psi(x_1, x_2) = N(x_1 + x_2) \exp\left[-\frac{1}{2} \alpha \left(x_1^2 + x_2^2\right)\right]$$

(a) [8] What is the correct value of the normalization constant $N$?

(b) [8] What is the probability that if we measure the position of one of the particles, it will be positive, $x_1 > 0$?

(c) [9] What is the probability that if we measure the position of both of the particles, they will both be positive, $x_1, x_2 > 0$?

5. A hydrogen atom is initially in a quantum state

$$|n, \ell, m_\ell, m_s\rangle = |2,1,0,\frac{1}{2}\rangle$$

where $n, \ell, m_\ell$, and $m_s$ are the quantum numbers associated with the Hamiltonian $H$, the total orbital angular momentum $L^2$, the $z$-component of the orbital angular momentum $L_z$, and the $z$-component of the electron spin $S_z$ respectively. Then, in rapid succession, the following quantities are measured: $L^2, L_z, J_\ell, J_z, L^2$, and $L_z$, where $J = L + S$ is the total (orbital plus spin) angular momentum of the electron.

(a) [4] What will be the possible outcomes of the first measurement of $L^2$ and $L_z$?

(b) [8] What will be the possible outcomes of the measurements of $J_z$ and $J^2$?

(c) [13] What will be the possible outcomes of the second measurements of $J_z, L^2$ and $L_z$?

Possibly Useful Integral:

$$\int_{-\infty}^{\infty} x^n \exp(-\beta x^2) dx = \frac{1}{2} \Gamma \left(\frac{n+1}{2}\right) \beta^{-\left(n+1\right)/2}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma(1) = 1, \quad \Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \sqrt{\pi}, \quad \Gamma(2) = 1, \quad \Gamma\left(\frac{5}{2}\right) = \frac{3}{4} \sqrt{\pi}, \quad \Gamma(3) = 2.$$