Quantum Mechanics Summer, 2012
Graduate Exam

Each question is worth 25 points. The points for individual parts are marked in square brackets. Some useful integrals are given at the end of the exam. To ensure full credit, show your work. Do any four (4) of the following five (5) problems. If you attempt all 5 problems you must clearly state which 4 problems you want to have graded.

1. A particle of mass \( m \) lies in an infinite square well with allowed region \( 0 < x < a \). At \( t = 0 \), the wave function is given by

\[
\Psi(x, t = 0) = \begin{cases} 
\sqrt{2/a} & \text{for } 0 < x < \frac{1}{2}a , \\
0 & \text{for } \frac{1}{2}a < x < a .
\end{cases}
\]

(a) [3] Will the particle remain localized in the left half of the well at later times?
(b) [7] The energy of the particle is measured. What is the probability that the result is the ground state energy \( E_1 \)? What is the probability that the result is the first excited state energy \( E_2 \)?
(c) [7] Calculate the probability that the energy yields the \( n \)'th state \( E_n \).
(d) [8] Show that the sum of the probabilities you found in part (c) is 1. The formula below may be useful.

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}
\]

2. Charmonium is a bound state of a charmed quark and a charmed anti-quark, each of which are spin \( \frac{1}{2} \) particles. The total spin \( S \) is the sum of the spins of the two quarks, \( S = S_1 + S_2 \) and the total angular momentum \( J \) is the sum of the orbital angular momentum and the total spin of the quark and anti-quark \( J = L + S \).

(a) [8] The lowest energy states of charmonium have no angular momentum, \( l = 0 \). Predict the possible values for the total spin quantum number \( s \), and the possible quantum numbers \( j \) associated with the total angular momentum of these states, together with the corresponding value of \( S^2 \) and \( J^2 \).
(b) [9] The next lowest energy states have \( l = 1 \). Predict the possible values for the total spin quantum number \( s \), and the possible quantum numbers \( j \) associated with the total angular momentum of these states, together with the corresponding value of \( S^2 \) and \( J^2 \).
(c) [8] One of the states in part (b) has \( J_\zeta \) measured, and it is found to have a value of \( +2\hbar \). For this state, what, if anything, can you predict about the value of \( S_\zeta \), \( L_\zeta \), \( S^2 \), \( L^2 \), and \( J^2 \), if they were measured for this state?
3. A particle of mass $m$ lies in a symmetric 2D harmonic oscillator with Hamiltonian

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2} m\omega^2 x^2 + \frac{1}{2} m\omega^2 y^2$$

In addition, there is a small perturbation

$$W = \gamma L_z = \gamma \left(xp_y - yp_x\right)$$

(a) [4] Find the eigenstates and energies of the unperturbed Hamiltonian. You do not need to give explicit forms for these eigenstates, it is sufficient to simply label them as, say, $|n_x, n_y\rangle$. Check that the ground state is non-degenerate, but the first excited state is degenerate.

(b) [9] Show that the ground state is an exact eigenstate of the perturbed Hamiltonian.

(c) [12] Find the energies of the first excited states to first order in $\gamma$, and the corresponding eigenstates to leading order.

Note: For the 1d harmonic oscillator with mass $m$ and angular frequency $\omega$, we have

$$x = \sqrt{\frac{\hbar}{2m\omega}} \left(a + a^\dagger\right) \quad \text{and} \quad p = i\sqrt{\frac{\hbar m\omega}{2}} \left(a^\dagger - a\right).$$

4. A particle of mass $m$ in three dimensions lies in the spherical infinite square well,

$$V(r) = \begin{cases} 0 & \text{if } r < a, \\ \infty & \text{if } r > a. \end{cases}$$

We are going to search for exact eigenstates of this potential of the form (for $r < a$):

$$\psi(r) = \frac{N \sin(kr)}{r}.$$

(a) [3] What do boundary conditions tell us about the value of $k$?

(b) [6] Demonstrate that this is, in fact, an eigenstate of the Hamiltonian, and determine its energy.

(c) [6] Determine the correct normalization factor $N$, possibly as a function of $k$.

(d) [10] At $t = 0$, the wave function takes the form

$$\Psi(r, t = 0) = \frac{2}{r\sqrt{5\pi a}} \sin^3\left(\frac{\pi r}{a}\right).$$

Find the wave function at all times.

Possibly helpful formulas: \[ \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin(3\theta) \]

\[ \nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \psi}{\partial \phi^2}. \]
5. A particle of mass $m$ in one dimension is trapped in the delta-function potential $V(x) = -\lambda \delta(x)$. This has exactly one bound state, given by the normalized wave function

$$\psi(x) = \sqrt{\alpha} e^{-\alpha |x|}.$$

(a) [16] Verify that this is a solution of Schrödinger’s equation, both away from the origin and at the origin, and determine an equation for $\alpha$ and the energy $E$.

(b) [9] At $t = 0$, the potential suddenly doubles in strength, so now $V(x) = -2\lambda \delta(x)$.

Determine the probability that the particle remains bound.

Possibly Useful Integrals:
For each of the following integrals, $m$ and $n$ are assumed to be positive integers.

$$\int_{0}^{a} \sin\left(\frac{\pi nx}{a}\right) \sin\left(\frac{\pi mx}{a}\right) dx = \frac{a}{2} \delta_{nm}$$
$$\int_{0}^{a} \cos\left(\frac{\pi nx}{a}\right) \cos\left(\frac{\pi mx}{a}\right) dx = \frac{a}{2} \delta_{nm}$$
$$\int_{0}^{a} x \sin\left(\frac{\pi nx}{a}\right) dx = \frac{a^2}{2\pi n} (-1)^{n-1}$$
$$\int_{0}^{a} x^2 \sin\left(\frac{\pi nx}{a}\right) dx = \frac{2a^3}{\pi^3 n^3} \left[(-1)^n - 1\right] - \frac{a^3}{n} (-1)^n$$
$$\int_{0}^{a} x \cos\left(\frac{\pi nx}{a}\right) dx = \frac{a^2}{\pi^2 n^2} (-1)^n$$
$$\int_{0}^{a} x^2 \cos\left(\frac{\pi nx}{a}\right) dx = \frac{2a^3}{\pi^2 n^2} + \frac{3a^3}{4\pi^2 n^2}$$
$$\int_{0}^{a} x\sin^2\left(\frac{\pi nx}{a}\right) dx = \frac{a^2}{4}$$
$$\int_{0}^{a} x^2 \sin^2\left(\frac{\pi nx}{a}\right) dx = \frac{a^3}{6} - \frac{a^3}{4\pi^2 n^2}$$
$$\int_{0}^{a} x\cos^2\left(\frac{\pi nx}{a}\right) dx = \frac{a^2}{4}$$
$$\int_{0}^{a} x^2 \cos^2\left(\frac{\pi nx}{a}\right) dx = \frac{a^3}{6} + \frac{3a^3}{4\pi^2 n^2}$$
$$\int_{0}^{\infty} x^n e^{-\beta x} dx = \frac{n!}{\beta^{n+1}}$$
$$\int_{0}^{\infty} x^2 \sin\left(\frac{\pi nx}{a}\right) \cos\left(\frac{\pi mx}{a}\right) dx = -\frac{a^3}{2\pi n}$$