Please solve 3 of the following 4 problems.

1. The Earth of mass $M_e$ interacts with a distant object (Sun or Moon) of mass $M$ via gravity. Consider the gravitational force on a small particle of mass $m \ll M_e$ located on the surface of the Earth as shown in the figure below. In this figure, both objects of mass $m$ and $M$ are treated as point particles. We define the important relative coordinates between the three particles of masses $m$, $M_e$, and $M$ relative to an inertial frame (shown in the figure) in the following way:

$$\vec{r} = \vec{r}_1 - \vec{r}_2, \quad \text{[vector from } M_e \to m]$$
$$\vec{d} = \vec{r}_1 - \vec{r}_3, \quad \text{[vector from } M \to m]$$
$$\vec{R} = \vec{r}_2 - \vec{r}_3, \quad \text{[vector from } M \to M_e]$$

a) Write down the equations of motion for both $m$ and $M_e$ due to the gravitational forces placed on them by the other pair of objects. Make use of the unit vectors: $\hat{r}, \hat{d}, \hat{R}$.

b) Using these two equations, find an equation of motion for the relative coordinate $\vec{r} = \vec{r}_1 - \vec{r}_2$.

c) Confining your diagram and system to the equatorial plane, i.e. $m$ can only be on the Earth’s equator, draw a force diagram based on your expression from part (b). Consider the 4 points of the compass in your equatorial plane, where $m$ would be placed at 0, 90, 180, and 270 degrees relative to the Earth-$M$ axis.

d) Estimate the relative contribution of the Moon vs. the Sun to tidal forces on Earth. (Earth: radius 6378km; Sun: distance $1.496 \times 10^8$km, mass $1.989 \times 10^{30}$kg; Moon: distance $3.844 \times 10^5$km, mass $7.349 \times 10^{22}$kg)
2. A bead of mass $m$ is constrained to move on a circular hoop of radius $R$. The hoop rotates with constant angular velocity $\Omega$ around a fixed vertical diameter of the hoop, parallel to the uniform gravitational field.

a) Calculate the Lagrangian for this system in appropriate variables.

b) Derive Lagrange’s equation of motion.

c) Write down the Hamiltonian in terms of the canonical momenta and coordinates.

d) Obtain all of the Hamiltonian equations of motion.

e) Find the critical angular velocity $\Omega = \Omega_{\text{crit}}$ below which the bottom of the hoop is the only stable equilibrium position for the bead.

f) Find the stable equilibrium position when $\Omega > \Omega_{\text{crit}}$.

3. A billiard ball of radius $R$ and mass $M$ is struck with a horizontal cue stick held at a height $h$ above the billiard table.

a) The cue stick delivers an impulse that can be approximated by a constant force $F$ over a short time interval $\Delta t$. What is the change in linear momentum delivered by this impulse?

b) Given the moment of inertia of a sphere is $\frac{2}{5}MR^2$, find the value of $h$ for which the ball will roll without slipping. Assume the billiard table is a frictionless surface.

4. A uniform bar of mass $M$ and length $2\ell$ is suspended from one end by a spring of force constant $k$. The bar can swing freely only in the $x,y$ plane, and the spring is constrained to move only in the vertical (ie, $y$) direction.

a) Set up the equations of motion in the Lagrangian formulation. (You do not need to solve them.)

b) Determine the normal mode frequencies of oscillation.