Please solve 3 of the following 4 problems. Make sure it is clear which 3 problems you are selecting. Calculators are not allowed.

1. A particle of mass $m$ moves under the influence of a potential

$$V = -V_0 e^{-\alpha r}.$$ 

The constant $\alpha$ is positive and the particle has angular momentum of magnitude $L$ and energy $E < 0$.

(a) What is the force on the particle?
(b) Give expressions for the maximum and minimum radii of the orbit. It is sufficient to simply set up the equations which must be solved.
(c) What is the required radius of circular orbits? Again set up the equation.
(d) Show that the circular orbit is stable and that the ratio of the period of small oscillations about this orbit and the period of revolution is given by $(3 - \alpha R)^{-1/2}$, where $R$ is the radius of the circular orbit.

2. Two masses, $m_1$ and $m_2$ are placed at the ends of massless rods of length $l$ to create a pair of pendula free to swing in the vertical plane connecting them. The angles of the pendula are as shown so that a positive angle denotes displacement of one mass away from the opposing pendulum. At the midpoint of each rod is attached one end of a spring of spring constant $k' = 4k$. The spring has an unstretched length equal to the separation between the rods where they attach to the ceiling.

(a) What is the Lagrangian for the system?
(b) Determine the eigenfrequencies for the normal modes of vibration of the system.
(c) Determine the normal modes of vibration of the system.
3. Consider a mass, \( m \), that is subject to a spring force \( F = -kr \), while constrained to move on the surface of a cylinder of radius \( R \).

(a) Write down the Hamiltonian for the point mass, \( m \), subject to the spring force and subject to the constraint that its motion is confined to the surface of the cylinder of radius \( R \).

(b) Use Hamilton’s equations to derive the equations of motion for a point mass, \( m \), that is subject to a spring force \( F = -kr \), while constrained to move on the surface of a cylinder of radius \( R \).

(c) State conserved quantities, if any.

(d) Solve the equation(s) of motion.

4. A container, partially filled with water, accelerates as it slides down a ramp inclined at an angle \( \alpha \) relative to the horizontal. The coefficient of kinetic friction between the container and the ramp is \( \mu \). After some initial transient motion, the surface of the water in the container eventually reaches equilibrium, and is found to maintain a constant angle \( \theta \) relative to the horizontal.
(a) What is the acceleration $\vec{a}$ of the container in terms of $\mu$, $\alpha$, and $g$? Write the acceleration vector in terms of the unit vectors $\hat{i}$ and $\hat{j}$ shown in the figure.

(b) What is the equilibrium angle $\theta$ of the surface of the water? Hint: For fluids the surface is always perpendicular to any acceleration different from freefall.

c) What is the angle $\theta$ when $\mu \to 0$?