Quantum Mechanics Graduate Exam

Quantum Mechanics Summer, 2015

Each problem is worth 25 points. The points for individual parts are marked in square brackets. Some useful formulas are given at the end of the exam. To ensure full credit, show your work. Do any four (4) of the following five (5) problems. If you attempt all 5 problems you must clearly state which 4 problems you want to have graded.

1. Consider the operator describing the spin along an arbitrary direction $\theta$ in the $x$-$z$ plane measured from the $z$-axis:

$\hat{S}_\theta = \frac{\hbar}{2} \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$

(a) [3] What are the eigenvalues of this operator?

(b) [6] Show that its eigenvectors are $\begin{pmatrix} \cos \left(\frac{1}{2} \theta\right) \\ \sin \left(\frac{1}{2} \theta\right) \end{pmatrix}$ and $\begin{pmatrix} \sin \left(\frac{1}{2} \theta\right) \\ -\cos \left(\frac{1}{2} \theta\right) \end{pmatrix}$.

(c) [6] Consider a spin $\frac{1}{2}$ particle whose state vector is described by $|\psi\rangle = \begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix}$. If one were to measure the particle’s spin along the $z$-axis, what would be the possible outcomes, and what would be the probability of obtaining each of these outcomes?

Consider the arrangement below with the first device oriented along $z$:

(d) [2] If the second device is oriented along the $z$-axis, what percentage of particles will have spin up vs. down after traversing it?

(e) [8] If the second device is oriented along the $x$-axis, what percentage of particles will have spin up vs. down after traversing it? Of those with spin up, what percentage will have up vs. down after traversing the third device when it is oriented along the $z$-axis?
2. A particle of mass $m$ is in a potential defined by $V(x) = \begin{cases} 0 & \text{if } 0 < x < a, \\ \infty & \text{otherwise}. \end{cases}$

Its wave function at $t=0$ is given in the allowed region by $\psi(x,0) = A \sin^5(\pi x/a)$.

(a) [13] Find the wave function $\psi(x,t)$ at all times $t$.

(b) [7] Calculate $A$ without doing the normalization integral.

(c) [5] What is the probability that an energy measurement yields $E_3$, the third lowest energy eigenvalue?

Hint: Expand $\sin^5 \theta = \left[ \frac{1}{2i} (e^{i\theta} - e^{-i\theta}) \right]^5$ and rewrite in terms of $\sin(5\theta)$, $\sin(3\theta)$, etc.

3. A particle of mass $m$ in 3D is in the potential $V(r) = \begin{cases} \frac{\pi^2 \hbar^2 \alpha}{ma^2 \sin^2(\pi r/a)} & \text{if } r < a, \\ \infty & \text{if } r > a. \end{cases}$

(a) [5] If $\alpha = 0$, the ground state wave function is $\phi_1(r) = N_1 r^{-1} \sin(\pi r/a)$. Show this is an eigenstate of the Hamiltonian, and find its energy and normalization $N_1$.

(b) [10] If $\alpha = 1$, the ground state wave function is $\phi_2(r) = N_2 r^{-1} \sin^2(\pi r/a)$. Show this is an eigenstate of the Hamiltonian, and find its energy and normalization $N_2$.

(c) [10] Suppose initially $\alpha = 0$ and the particle is in the ground state. If $\alpha$ is then increased to the value $\alpha = 1$, what is the probability it remains in the ground state if the transition is (i) adiabatic, (ii) sudden?

4. A particle of mass $m$ lies in the 2D harmonic oscillator with potential $V(X,Y) = \frac{1}{2} m \omega^2 \left( X^2 + Y^2 \right)$. To this is added a small perturbation $W = \delta m \omega^2 \left( X^2 + 4XY + Y^2 \right)$.

(a) [2] Label the exact eigenstates of the unperturbed Hamiltonian, and give their energies.

(b) [10] Determine the ground state eigenstate and its energy to first order in $\delta$.

(c) [13] For the first excited states, determine the eigenstates to leading order and the energies to first order in $\delta$.

5. Consider the three operators $A = P_y^2 - P_x^2$, $B = 2 P_x P_y$, and $L_z = X P_y - Y P_x$.

(a) [8] Work out the three commutators of these operators with each other. You may find it helpful to recall that $[L_z, P_x] = i \hbar P_y$ and $[L_z, P_y] = -i \hbar P_x$.

(b) [9] Based on the results of part (a), deduce two non-trivial inequalities relating the uncertainties of these operators and their expectation values.

(c) [8] Show that if you are in an eigenstate of $L_z$, then $\langle A \rangle = \langle B \rangle = 0$. 

Possibly Useful Formulas:

1D harmonic oscillator: \( X = \sqrt{\frac{h}{2m\omega}} (a + a^\dagger) \), \( a\,|n\rangle = \sqrt{n}\,|n-1\rangle \), \( a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \)

Trigonometry:
\[
\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta ,
\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta .
\]

Laplacian in spherical coordinates:
\[
\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} .
\]

Integrals:
\[
\int_0^a \sin(\pi x/a) \,dx = \frac{2}{\pi} a , \quad \int_0^a \sin^2(\pi x/a) \,dx = \frac{1}{2} a , \quad \int_0^a \sin^3(\pi x/a) \,dx = \frac{4}{3\pi} a ,
\int_0^a \sin^4(\pi x/a) \,dx = \frac{3}{8} a , \quad \int_0^a \sin^5(\pi x/a) \,dx = \frac{16}{15\pi} a , \quad \int_0^a \sin^6(\pi x/a) \,dx = \frac{5}{16} a
\]