Please solve 3 of the following 4 problems.

1. A solid half-sphere of mass \( M \) and radius \( R \) is shown placed on a table as sketched in the figure on the left below. The sphere is initially pushed in the \( y-z \) plane in a manner indicated by the figure on the right below, and rolls without slipping. The body \( z \)-axis (labeled \( z' \)) has been rotated through an angle \( \theta \). The center of mass of the half sphere in the figure on the left is located a distance \( 5R/8 \) above the origin along the \( z \)-axis. The moment of inertia tensor component for rotation about an axis parallel to the \( x \)-axis, but through the center of mass is \( I_{11} = (83/320)MR^2 \).

   ![Diagram of half-sphere](image)

   a) What is the center of mass position vector of the object as a function of \( \theta \)?
   b) What is the center of mass velocity vector of the object as a function of \( \theta \) and \( \dot{\theta} \)?
   c) What is the Lagrangian of the system in the case where the rotation angle remains small?
   d) What is the frequency of small oscillations for the half-sphere? You might want to consider Lagrange’s equations of motion.

2. Two atoms in a diatomic molecule (masses \( m_1 \) and \( m_2 \)) interact through a potential energy function of the form

\[
U(r) = \frac{a}{4r^4} - \frac{b}{3r^3},
\]

where \( a \) and \( b \) are constants and \( r \) is the separation of the atoms.

   a) Find the equilibrium separation of the atoms and the frequency of small oscillations about equilibrium, assuming the molecule does not rotate. How much energy must be supplied to the molecule in equilibrium in order to break it up (that is, so that \( r \to \infty \))? 
   b) Now allowing the molecule to rotate, determine the maximum angular momentum which the molecule can have without it breaking up.
3. Three masses are arranged as shown in the figure. There is no friction between \( m_1 \) and \( M \), and no friction between \( M \) and the floor. \( m_1 \) and \( m_2 \) are connected by a massless rope that passes over a massless and frictionless pulley. Assume that the horizontal separation between \( M \) and \( m_2 \) remains constant (\( m_2 \) cannot swing out).

a) Construct the Lagrangian for the system. Clearly describe the generalized coordinates you use.

b) There is a conserved quantity for this system. Determine the quantity and show that it is conserved within the Lagrangian or Hamiltonian framework.

c) Solve the equations of motion for \( M \) and \( m_1 \) (describe the positions of \( M \) and \( m_1 \) as functions of time). Assume all masses are at rest at \( t = 0 \) and only consider the time until \( m_2 \) reaches the ground.

d) Use the method of Lagrange multipliers to find the tension in the string.

4. A projectile of mass \( m \) is launched at an angle of 45° from the horizontal with some initial kinetic energy, \( E_0 \). At the top of its trajectory, the projectile explodes into two fragments with masses \( m_1 \) and \( m_2 \) with \( m_1 + m_2 = m \). The explosion provides an additional energy \( E_0 \) to the system. The motion is still confined to the \( x - y \) plane, and the fragment of mass \( m_1 \) travels straight down with a speed \( v_1 \).

a) Find the components of the velocities for each of the two fragments.

b) What is the horizontal range for \( m_2 \) as measured from the initial launch position?

Express your answers in terms of \( m \), \( E_0 \), \( v_1 \), and the gravitational acceleration \( g \).