91.1 Equivalence Relations

Which of these objects are the same as $A$?

\[ A \quad B \quad C \quad D \quad E \quad F \quad G \quad H \quad I \quad J \]

It depends on what we mean by the same?

- congruent: $A, F, B$
- similar: $A, B, F, G$
- three-sided: $A, B, F, G, I$
- convex: $A - G, I$
- polygons: $A, B, C, E - J$
A binary relation $\mathcal{R}$ relates two objects in the same set.

For example, $<$ is a binary relation on the real numbers.

\[
2 < 3 \quad 3 \not< 2 \\
-5 < 0 \quad 2 \not< 2
\]

Ex: $\equiv$ on people where $a \equiv b$ if $a$ is a descendant of $b$

- me $\equiv$ my dad
- me $\equiv$ my grandmother
- my dad $\not\equiv$ me

Reflexive
Symmetric
Transitive

Equivalence relation

Ex: $\mathbb{Z}$ mod 10

Consider set $X = \{(a, b) : a, b \in \mathbb{Z}, b \neq 0\}$

Equality

$(a, b) \sim (c, d)$ if $ad = bc$

HW: Find 3 examples of equivalence relns
- at least 1 mathematical
- at least 1 not.
Consider the set $\mathbb{Z} \times \mathbb{Z} = \{(a,b) : a,b \in \mathbb{Z}\}$ and relation $(a,b) \sim (c,d)$ if $ad=bc$.

Is this an equivalence relation?
- Reflexivity: Yes, since $(a,a) \sim (a,a)$ for all $a \in \mathbb{Z}$.
- Symmetry: Yes, if $(a,b) \sim (c,d)$, then $(c,d) \sim (a,b)$.
- Transitivity: No, because $(1,1) \sim (0,0)$ and $(0,0) \sim (1,0)$, but $(1,1) \not\sim (1,0)$.

**HW**

1. Consider the following argument:
   - **Claim:** If relation $\sim$ on a set $X$ is both symmetric and transitive, then it is reflexive.
     - Why? $x \sim y$ implies $y \sim x$ by symmetry. Now apply the transitive property: since $x \sim y$ and $y \sim x$, we conclude $x \sim x$.
   - This claim is **FALSE**. What's wrong with the argument?

2. The reflexive, symmetric, and transitive properties do not imply one another. For each possibility, of them being true/false, find a relation with obeying that possibility.
   - i.e., $\mathbb{Z}$ $\sim$
     - True False False False
     - True True False False

**Defn**

Let $X$ be a set with equivalence relation $\sim$.

The **equivalence class** $[x]$ of element $x \in X$ consists of all elements of $X$ that are equivalent to $x$.

Frequently, we will discuss the set of equivalence classes, which we denote $X/\sim$.

e.g. skirt color

$X = \{\text{shirts}\}$

$X/\sim = \{\text{skirt colors}\}$
To understand equivalence relations on geometric objects, we often try to construct a map from one to the other. For example, similar triangles $\triangle T_1 \sim \triangle T_2$ which preserves the angles. If so, the triangles are similar. If not, they’re not.

**Def.** Let $X, Y$ be sets. A map (aka function) $f : X \to Y$ is an **injection** (or **one-to-one**) if distinct pts of $X$ must map to distinct points of $Y$.

\[\forall a, b \in X, \quad a \neq b \implies f(a) \neq f(b)\]

(contrapositive) $f(a) = f(b) \implies a = b$.

Next, $f$ is a **surjection** (aka **onto**) if $\forall y \in Y \exists x \text{ mapping to it}$, i.e., $f(x) = y$.

Finally, $f$ is a **bijection** if it is both an injection and a surjection.

**Ex.** $f(x) : \mathbb{R} \to [0, \infty)$

\[f(x) = x^2\]

- not an injection, since $f(-1) = f(1) = 1$
- is a surjection.

**Ex.** $g(x) : \mathbb{R} \to \mathbb{R}$

\[g(x) = x^3\] is a bijection.

**Ex.** Find a bijection from $(-1, 1)$ to $\mathbb{R}$.

Recall that $\tan(\frac{\pi}{4}, \frac{\pi}{2})$ to all of $\mathbb{R}$.

Thus $\tan(\frac{\pi x}{2})$ maps $(-1, 1)$ to all of $\mathbb{R}$.

**Fact:** If a bijection from $X$ to $Y$ exists, then $X$ and $Y$ are the same size.

**(DX)** check this if either is finite.

**Warning:** there are different sizes of infinite sets: IN, Z, Q are “countable” \(\mathbb{R}\) is “uncountable”

**Ex.** Find a bijection from $\mathbb{R}$ to $[0, 1)$

- Idea: something must map to 0; try $\frac{1}{x}$
- send $\frac{1}{2} \to \frac{1}{2}$
- $\frac{1}{4} \to \frac{1}{3}$
- $\frac{1}{n} \to \frac{1}{n-1}$ (n>1)
- otherwise, send $x \notin \{\frac{1}{n}: n \in \mathbb{N}\}$ to itself.

This is not continuous.
Ex: bijection from unit circle $S^1 = \{(x,y): x^2+y^2=1\}$ to square \textit{armed by} max $(x,y) = 1$

Idea: push out radially

Ex: Find a subset of $\mathbb{R}^3$ and a bijection to the “congruence classes of triangles” — i.e., the equivalence classes of triangles under the (equivalence) relation of congruence.

Given triangle $T$ with side lengths $a, b, c$.

we may assume $a \leq b \leq c$.

we claim that any other triangle with these side lengths must be congruent to $T$. (DX - convince yourself)

Does any triple $(a, b, c)$ produce a triangle?

No — the side lengths must obey the \textbf{Triangle Inequality}.

\begin{align*}
& a + b > c \\
& b + c > a \\
& a + c > b
\end{align*}

Consider this set in $\mathbb{R}^3$:

$D = \{(a, b, c): 0 < a \leq b \leq c < a+b\}$

Viewing a geometric object as a single point in the space of all such objects leads to the modern idea of \textit{configuration spaces}.

\textbf{Research breakthrough (2011)} The space of all $n$-sided polygons of fixed perimeter (say $P$) has a natural bijection with the space of all oriented planes in $\mathbb{C}^n$, a well-studied space known as a \textit{Stiefel manifold}.

(Continiello, DePulu, Shankwiler).
§1.2 Bijections

Motivation

Injection, surjection

\[ 0,1 \to \mathbb{R} \quad f = \tan x \]
\[ g = \tan \left( \pi x - \frac{\pi}{2} \right) \]

\[ (0,1) \to [0,1) \quad \text{harder} \]

- \[ S^1 \to \mathbb{R}^2 \quad \text{square in} \]

Cartesian product \[ A \times B = \{ (a,b) : a \in A, b \in B \} \]

Topologically \((0,1)\) and \([0,1)\) are very different (open/closed)
so bijections don't tell us precisely what we want.
We need something stronger (continuity).

Examples 1) subset of \(\mathbb{R}^3\) and \(\frac{3}{2} \text{Triangles} \)

[in notes]

where 2 triangles are equivalent if congruent

congruence classes

2) \[ \Theta \to S^1 \times S^1 \quad \text{torus} \]

[longitude, meridian]
**Definition**

Composition

\[ g \circ f (x) = g(f(x)) \]

**Identity map**

\[ id_X \]

**Inverse map**

\[ g \circ f = id_X \]

\[ f \circ g = id_Y \]

\[ f: X \to Y \]

\[ g: Y \to X \]

\[ g = f^{-1} \]

**Theorem**

\[ f: X \to Y \text{ is a bijection} \iff f^{-1} \text{ exists.} \]
9.3 Continuity

When is \( f : X \to Y \) cts?

In calc I = cts -- pencil not lifting.

\( f \) is cts at \( a \) if

\[ \forall \varepsilon > 0 \ \exists \delta > 0 \ \text{s.t.} \]

\[ |x - a| < \delta \]

then \( |f(x) - f(a)| < \varepsilon \)

i.e., points that are close together stay close together.

In general, \( f : X \to Y \)

\[ \text{same defn} \]

\( f \) is continuous at \( a \in X \) if

\[ \forall \varepsilon > 0 \ \exists \delta > 0 \ \text{s.t.} \]

\[ \text{if distance} \ (x, a) < \delta \ \text{then distance} \ (f(x), f(a)) < \varepsilon \]

metric
**Wend fact:** If non-metrizable spaces (see math 731)

**Examples**

\[ f: (0, 1) \to [0, 1) \]

Disjoint at each \( x = \frac{1}{n} \) \((n \geq 2)\)

**Ex:**

\[ S^1 \to S^1 \]

Unit circle.

\[ z \to z^2 \]

\[ x + iy \to \]

In polar coords,

\[ z = re^{i\theta} \quad \Rightarrow \quad S^1 = \text{unit circle, so } r = 1 \]

\[ e^{i\theta} \]

\[ z^2 = (e^{i\theta})^2 = e^{2i\theta} \]

Geometrically this is cts.

If we measure distance on \( S^1 \) via \( \Theta \)

To make \( d(s(z), s(z_0)) < \varepsilon \), choose \( z \) s.t.

\[ d(z, z_0) < \delta = \varepsilon \]
Ex: Area function $A(x)$ for planar region $R$ of finite area

If $R$ is bounded, put $x=0$ on left side
and keep $R$ between $y=0$ and $y=b$.

Moving $S$ in $x$ can add at most $b \cdot S$
so, to make $b \cdot S < \varepsilon$, we want $S < \frac{\varepsilon}{b}$.

If $R$ is unbounded in $y$, we must be more careful.
(If $R$ is bounded in $y$ + unbounded in $x$, above plan works fine).

Idea: since $R_{\text{up}}$ has finite area
by drawing $y=a_1, y=a_2, y=a_3, \ldots$
we eventually contain most of the area of $R$.

Given $\varepsilon$, we can draw some $y=b$ such that
less than $\frac{\varepsilon}{2}$ of area of $R$ is above $y=b$.
(Actually showing this is a pain.)

Then by moving in $x$, we may add some of $R_i$
and at most $2b \cdot S$

$$A(x) - A(x_0) \leq 2b \cdot S + \text{Area}(R_i)$$
$$\leq 2b \cdot S + \frac{\varepsilon}{2}$$
1.4 Topological Equivalence

**Motivation** We have talked about equivalence relations — a way of saying that two objects are "the same" (with respect to that relation). For example,

- Congruence on the set of triangles tells us when triangles are the same size.
- Similarity on the set of triangles tells us when triangles have the same angles.
- Bijection on sets tells us when sets have the same size.
- In topology, what guarantees that two objects are the same? i.e., all topological properties are the same.

It should be unsurprising that continuity is important.

**Defn** A homeomorphism \( h : X \rightarrow Y \) is a bijection from \( X \) to \( Y \) with both \( h \) and \( h^{-1} \) continuous.

We say that \( X \) and \( Y \) are **homeomorphic** if there exists a homeomorphism between them.

n.b., to talk about continuity, it seems like we need a notion of distance (i.e., a metric). Sets that have a metric are called **metric spaces**.

**3 Weird Facts**

1. Sets may have a lot of different metrics, different enough that the idea of continuity changes.
2. There are sets that do not admit any metrics.
3. On these sets, we may still define continuity (by specifying what open + closed sets are). Once we do so, we call it a **topological space**. All metric spaces are topological spaces, but not vice-versa.

We won't work with any of the weird sets in 3, or weird metrics in 1.

**Ex:** \( h : S^1 \rightarrow \text{square} \)

is a homeomorphism.

- Injection \( \checkmark \)
- \( h \) cts \( \checkmark \)
- \( h^{-1} \) cts \( \checkmark \)

**Proposition** Homeomorphic is an equivalence relation on metric spaces.

**Reflexive:** Identity map \( i : X \rightarrow X \) is a homeomorphism.

**Symmetric:** If \( h : X \rightarrow Y \) is a homeomorphism, consider \( g = h^{-1} : Y \rightarrow X \).

\[ h : \text{a bijection} \iff g = h^{-1} : \text{a bijection}. \]

\[ h \text{ cts.} \iff g^{-1} = (h^{-1})^{-1} = h^{-1} \text{ cts.} \]

Thus \( h^{-1} : Y \rightarrow X \) is a homeomorphism.

**Transitive:** If \( f : Y \rightarrow Z \) and \( g : Y \rightarrow Z \) are homeomorphisms, take the composition \( h = g \circ h : X \rightarrow Z \).

Composition of bijections is a bijection \( h^{-1} : Z \rightarrow X \)

Composition of cts maps is cts. \( h^{-1} = (g \circ f)^{-1} = f^{-1} \circ g^{-1} \)
Ex: \[ w: [0,2\pi) \to S^1 \text{ is a continuous bijection.} \]
\[ w(t) = (\cos t, \sin t) \]
However \[ w^{-1}: S^1 \to [0,2\pi) \] is not a bijection.

The points \((1,0)\) and \(p\) can be chosen arbitrarily close together on \(S^1\) but \(w^{-1}(1,0) = 0\) while \(w^{-1}(p)\) is close to \(2\pi\). \(\therefore w^{-1}\) is not cts. at \((1,0)\).

\(\therefore w\) is not a homeomorphism.

Topologically, a circle and an interval are fundamentally different. No homeomorphism between them exists.

Some notation

standard disk \(D^2\) is \(\{x_1^2 + x_2^2 \leq 1\} \subset \mathbb{R}^2\)

unit disk is anything congruent to it.

disk \((D^2, \mathbb{R}^2)\) is anything homeomorphic to it.

standard \(n\)-dim. ball \(B^n\) is \(\{x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1\} \subset \mathbb{R}^n\)

unit \(n\)-ball is anything congruent to it.

\(n\)-ball \(B^n\) is any set homeomorphic to it.

standard \(n\)-dim. sphere \(S^n\) is \(\{x_1^2 + x_2^2 + \cdots + x_n^2 + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}\). It is the boundary of the standard \((n+1)\)-ball.

unit \(n\)-dim. sphere is anything congruent to it.

\(n\)-sphere \(S^n\) is anything homeomorphic to it.

Example: Show that a circle \(S^1\) is homeomorphic to the trefoil knot below.

Parameterize \(S^1\) by \(t \in [0,2\pi]\) as usual.
Since all circles are homeomorphic to the unit circle, we may start with the unit circle.

- $h$ is a bijection
  - $h$ is continuous:
    - points on circle that are close, stay close
  - $h^{-1}$ is continuous:
    - ditto.

**Stereographic Projection**

**Fact**: $S^2 \setminus \{N\} \cong \mathbb{R}^2$

Draw $\mathbb{R}^2$ as the equatorial plane.

Define $T: S^2 \setminus \{N\} \rightarrow \mathbb{R}^2$:

- $T$ takes a point $p$ on the sphere to a point $T(p)$ — where the line from $N$ through $p$ intersects the plane.

**Today:**
- What is $T(S)$? South pole
- Where does $T$ send the equator?
  - The northern hemisphere?
  - The southern hemisphere?

Interpret where $T$ might send $N$?

**HW** show $T(x, y, z) = \left( \frac{x}{1 - z}, \frac{y}{1 - z}, 0 \right)$

$$T^{-1}(u, v, 0) = \left( \frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right)$$

**Hint:** Use cylindrical coordinates instead of $(x, y, z)$ and $(u, v, 0)$

- $x = r \cos \theta$
- $y = r \sin \theta$
- $z = z$
1.5 Topological Invariants

**Defn** a property that is preserved under homeomorphisms is a **topological invariant**.

**Ex:** \{1, 2, 3\} and \{1, 3, 4\} are not homeomorphic.

Size/cardinality of a set is preserved via bijections i.e. iso a top inv.

**Ex:** \(\mathbb{N}\) and \(\mathbb{R}\) are not homeomorphic.

**Defn** a path from \(a\) to \(b\) in \(X\) is a cts. map \(\alpha: [0,1] \to X\) with \(\alpha(0) = a\) , \(\alpha(1) = b\).

The path component of \(a\) is the set of all points connected to \(a\) via a path.

- Two in same path comp is equiv rel.
- \(X\) is path-connected if all points of \(X\) lie in the same path component; i.e., if \(X\) only has 1 path comp.

**Theorem 1.44** Suppose \(\alpha: [0,1] \to A \cup B\) is a path with \(\alpha(0) \in A\), \(\alpha(1) \in B\).

Then 3 seq of points in \(A\) converging to a point in \(B\) or
3 seq of points in \(B\) converging to a point in \(A\).

**Ex:**

![Path Components Diagram]

Proving this requires some facts from analysis (upper bounds, completeness) — read this, but we don’t need it.

**Ex:** \([1, \infty) \cup (0,1]\) is not path connected

**(Ex)** We can connect all pts in \([0,1]\) to \(-1\). Thus they are all in the same path component.

We can also connect all pts in \((0,1]\) to \(1\). Thus they are all in the same path component.

We can use the theorem to show \(-1\) and \(1\) are not path connected. (We will argue by contradiction. We assume the opposite of what we want to show.)

What should \(A\) be? \(B\)?

\(A = [1, \infty)\)

\(B = (0,1]\)

**Suppose** \(-1\) and \(1\) are path connected. Then, there is some path \(\alpha: [0,1] \to X = A \cup B\) s.t.

\(\alpha(0) = -1\), \(\alpha(1) = 1\).

Then the theorem implies either

1) 3 sequence in \(A = [1, \infty)\) converging to a point in \(B = (0,1]\), \(\gamma\), to a positive value

2) 3 sequence in \(B = (0,1]\) converging to a point in \(A = [1, \infty)\), \(\epsilon\), to a negative value

**Q:** Take a convergent sequence \((a_n)_{n=1}^{\infty}\) with all terms positive. Must it converge to a positive number?

No. \(\frac{1}{n} \to 0\). But its limit is either positive or \(0\); it cannot converge to a negative value.

So (2) cannot happen. Similarly (1) cannot happen.
Thus, either our theorem is false (it not) or our initial assumption must be false.

We may then conclude that there is no path from -1 to 1 in X.

\[ X \] has 2 path components.

\[ \text{Ex: Any interval } (a, b) \cup (b, c) \text{ is not path connected.} \]

\[ \text{Theorem} \quad \text{Let } f: X \rightarrow Y \text{ be continuous. Then } f \text{ maps a path component of } X \text{ into some path comp of } Y. \]

\[ \text{n.b., not necessarily onto} \]

\[ \text{Proof:} \quad \text{Take } a, b \text{ in some path comp of } X. \]

\[ \text{We must show } f(a), f(b) \text{ lie in the same path comp of } Y. \]

\[ \text{There is some path } \gamma: [0, 1] \rightarrow X \text{ from } \gamma(0) = a \text{ to } \gamma(1) = b. \]

\[ \text{Then consider the composition} \]

\[ f \circ \gamma: [0, 1] \rightarrow X \rightarrow Y \]

\[ \gamma(0) \rightarrow f(\gamma(0)) \]

Is it cts? \quad Yes \quad \text{composition of cts fns is cts.} \]

\[ \text{It maps } 0 \text{ to } f(\gamma(0)) = f(a), \]

\[ \text{and } 1 \text{ to } f(\gamma(1)) = f(b). \]

\[ \Box \]

\[ \text{Ex: (DX) Determine the path components of } \mathbb{R}^2 \setminus S^1. \]

\[ \text{Answer: } 2 \text{ path components: } (1) \text{ inner disk } + (2) \text{ points outside} \]

\[ \text{To show it,} \]

\[ (1) \text{ find a path connecting any 2 points of } A \]

\[ \text{use a straight line} \]

\[ (2) \text{ find a path connecting any 2 points of } B \]

\[ \text{can we always use a straight line?} \]

\[ \text{If we cannot, first travel around a concentric circle to } S^1 \text{ containing } a \text{ until you reach the polar angle of } b. \]

\[ \text{Call this path } \gamma_s: [0, 1/2] \rightarrow B \]

\[ \text{Then if necessary move radially to reach } b \text{ - call this} \]

\[ \gamma_c: [1/2, 1]. \]

\[ \text{We then } \gamma \text{ as } \gamma(t) = \begin{cases} \gamma_s(t) & \text{if } t \in [0, 1/2] \\ \gamma_c(t) & \text{if } t \in [1/2, 1] \end{cases} \]

\[ \text{Defn. A subset } Y \text{ of a path-connected set } X \text{ separates } X \text{ if } X \setminus Y \text{ is not path-connected.} \]

Above, \( S^1 \) separates \( \mathbb{R}^2 \).
The general version of this example is much harder... that any curve homeomorphic to $S^1$ separates $\mathbb{R}^2$. It's known as the **Jordan Curve Theorem**, named after Camille Jordan (French, 1887). His proof was incomplete... full proof by Veblen (Princeton, 1912).

**Notation**

$C_a =$ path-compt of $a$ in $X$

$P(X) = \{ \text{path compts of } X \}$

**Corollary** Any homomorphism $h: X \to Y$ induces a bijection $h_\#: P(X) \to P(Y)$ on the path components. In particular, homeomorphic spaces must have the same number of path components.

**Proof:** First, $h_\#$ is a well-defined map, since by the theorem, all pts of path compct $C_x$ of $X$ land in the path compct $C_{h(x)}$ of $Y$. So $h_\#(C_x) = C_{h(x)}$.

We show $h_\#$ is a bijection:

- $h_\#$ is a surjection: Consider a path component of $Y$ containing $r$, i.e., path compct $C_r$.
  
  Since $h$ is onto, $\exists x \text{ s.t. } h(x) = r$. By the theorem, $h$ must map all points of $C_x$ into $C_r$.

  \[
  \begin{array}{c}
  X \\
  \xrightarrow{h} \\
  Y
  \end{array}
  \]

  Thus $C_r = h_\#(C_x)$.

  : $h_\#$ is onto.

- $h_\#$ is an injection: For $x, y \in X$, suppose $C_x$ and $C_y$ both map via $h_\#$ to $C_r$, i.e.,
  
  $h_\#(C_x) = h_\#(C_y) = C_r$. Consider $h^{-1}$ which is cts since $h$ is a homeo.
  
  Since $h^{-1}(h(x)) = x \in C_x$, $h^{-1}$ maps $C_r$ into $C_x$.
  
  Since $h^{-1}(h(y)) = y \in C_y$, $h^{-1}$ maps $C_r$ into $C_y$.

  By the theorem, $C_x$ and $C_y$ must be the same path compct.

  : $h_\#$ is 1-1, and thus it's a bijection.

Since bijections preserve sizes of sets, $P(X)$ and $P(Y)$ have the same cardinality, i.e., $X$ and $Y$ have the same number of path compts.
Example: (important!) Show that $(0,1)$ and $[0,1)$ are not homeomorphic.

We found a bijection between them in §1.2, but it was not continuous! But this means they have the same cardinality, so we cannot use that invariant.

Let's try path-connectedness. Are both path-connected? Yes.
We only know a few invariants. Should we give up? No! Let's use separability into path components.

Removing the point $0 \in (0,1)$ does not separate it.
If $h : (0,1) \to (0,1)$ were a homeo., this means $h(0)$ should not separate $(0,1)$.

Q: which points of $(0,1)$ separate it?
   → All of them. So $h(0)$ does not separate $(0,1)$ for any map $h : [0,1) \to (0,1)$

: No such $h$ is a homeomorphism
: $[0,1) \neq (0,1)$

Example: $T^2 \neq S^2$

Draw a closed curve on $S^2$ — it separates $S^2$. (DX)

Since there are closed curves on the torus that separate it, while no such curves exist on the sphere, we see the torus and sphere are not homeomorphic.
Not all homeomorphisms $h: X \to Y$ can be obtained by deforming $X$.

Example
**Defn** Let $A, B$ be subsets of $X$. An **ambient isotopy in $X$ from $A$ to $B$** is a map $h : X \times [0,1] \to X$ such that, if $h(x, t)$ is denoted $h_t(x)$,

1. $h_t : X \to X$ is a homeomorphism for each $t \in [0,1]$
2. $h_0(x)$ is the identity map
3. $h_1(A) = B$, i.e., $h_1$ sends $A$ to $B$

By the end of the isotopy, $A$ has been deformed to $B$. 
we can think of time as going from 0 to 1.

we ask that it be continuous.

since we move by changes slightly, we have a homomorphism $\phi: \Delta \to X$.

at each time $t$, we have a homemo $h_t: A \to B$.

we start at $A \otimes X$.

we move through time by changes $X$ slightly.

at every time, did we still have a triangle? yes.

what was important for triangles? homos vs deformations.

Ambient isotopy.

Ex: Dehn twist.

Ex: Deform any triangle to any other.

For Fri.

Homos are not homos.

And spaces that are homos.

To your assignment.