$\mathbb{Z}$-graded noncommutative projective geometry

Algebra Seminar

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Overview

1 Preliminaries
   Pre-talk catchup
   Noncommutative things

2 Noncommutative projective schemes

3 \( \mathbb{Z} \)-graded rings
   Prior work
   Generalized Weyl algebras

4 Future direction

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What you missed in the pre-talk

- Throughout, $\mathbb{k} = \bar{\mathbb{k}}$, $\text{char}(\mathbb{k}) = 0$
- $\Gamma$ an abelian group
- A $\Gamma$-graded $\mathbb{k}$-algebra $A$ has $\mathbb{k}$-space decomposition
  \[ A = \bigoplus_{\gamma \in \Gamma} A_{\gamma} \]
  such that $A_\gamma A_\delta \subseteq A_{\gamma + \delta}$
- A graded right $A$-module $M$:
  \[ M = \bigoplus_{\gamma \in \Gamma} M_\gamma \]
  such that $M_\gamma \cdot A_\delta \subseteq M_{\gamma + \delta}$
What you missed in the pre-talk

- \( A \) graded by \( \mathbb{N} \) (or \( \mathbb{Z} \)), the graded module category \( \text{gr-}A \):
- Objects: finitely generated graded right \( A \)-modules
- Hom sets (degree 0) graded module homomorphisms:

\[
\text{hom}_{\text{gr-}A}(M, N) = \{ f \in \text{Hom}_{\text{mod-}A}(M, N) \mid f(M_i) \subseteq N_i \}
\]

- The shift functor:

\[
S^i : \text{gr-}A \rightarrow \text{gr-}A
\]

\[
M \mapsto M \langle i \rangle
\]

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The Picard group

- For a category of graded modules (e.g. $\text{gr}-A$) we define the **Picard group**, $\text{Pic}(\text{gr}-A)$ to be the group of autoequivalences of $\text{gr}-A$ modulo natural isomorphism.

- (If $R$ is a commutative $k$-algebra, $\text{Pic}(\text{mod}-R) \cong \text{Pic}(R)$, the isomorphism classes of invertible unitary $R$-bimodules)

- For a noncommutative ring, $A$, $\text{Pic}(\text{gr}-A)$ can be nonabelian.
Morita equivalence

- Two rings $R$ and $S$ are called **Morita equivalent** if $\text{mod-}R$ is equivalent to $\text{mod-}S$ (if and only if $\text{R-mod equivalent to S-mod}$).
- For $R$ and $S$ commutative, then $R$ is Morita equivalent to $S$ if and only if $R \cong S$.

**Example**

$R$ a ring, then $M_n(R)$ (the ring of $n \times n$ matrices) is Morita equivalent to $R$.

- $R$ and $S$ are **graded Morita equivalent** if $\text{gr-R}$ is equivalent to $\text{gr-S}$.
Noncommutative is not commutative

- Commutative graded ring: $R \leftrightarrow \text{Proj } R$
- Noncommutative ring: not enough (prime) ideals

<table>
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<th>The Weyl algebra</th>
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<td>$\mathbb{k}\langle x, y \rangle / (xy - yx - 1)$</td>
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<td>is a noncommutative analogue of $\mathbb{k}[x, y]$ but is simple.</td>
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<th>The quantum polynomial ring</th>
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<td>The $\mathbb{N}$-graded ring</td>
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<tr>
<td>$\mathbb{k}\langle x, y \rangle / (xy - qyx)$</td>
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<td>is a “noncommutative $\mathbb{P}^1$” but for $q^n \neq 1$ has only four homogeneous prime ideals (namely $(0), (x), (y), \text{ and } (x, y)$).</td>
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Sheaves to the rescue

• The Beatles (paraphrased):

  “All you need is sheaves.”

• Idea: You can reconstruct the space from the sheaves.

Theorem (Rosenberg, Gabriel, Gabber, Brandenburg)
Let $X$, $Y$ be quasi-separated schemes. If $\text{qcoh}(X) \equiv \text{qcoh}(Y)$ then $X$ and $Y$ are isomorphic.
Sheaves to the rescue

• All you need is modules

Theorem

Let $X = \text{Proj } R$ for a commutative, f.g. $k$-algebra $R$ generated in degree 1.

1. Every coherent sheaf on $X$ is isomorphic to $\tilde{M}$ for some f.g. graded $R$-module $M$.

2. $\tilde{M} \cong \tilde{N}$ as sheaves if and only if there is an isomorphism $M \geq n \cong N \geq n$.

• Let $\text{gr-}R$ be the category of f.g. $R$-modules. Take the quotient category

$$q\text{gr-}R = \text{gr-}R/\text{fdim-}R$$

• The above says $q\text{gr-}R \equiv \text{coh}(\text{Proj } R)$.

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Noncommutative projective schemes

• A (not necessarily commutative) connected graded \( k \)-algebra \( A \) is

\[
A = k \oplus A_1 \oplus A_2 \oplus \cdots
\]

such that \( \dim_k A_i < \infty \) and \( A \) is a f.g. \( k \)-algebra.

**Definition (Artin-Zhang, 1994)**

The noncommutative projective scheme \( \text{Proj}_{\text{NC}} A \) is the triple

\((\text{qgr}-A, \mathcal{A}, S)\)

where \( \mathcal{A} \) is the distinguished object and \( S \) is the shift functor.

• Idea: Use geometry to study \( \text{Proj}_{\text{NC}} A = \text{qgr}-A \) to study \( A \).
All you need is modules

- Can attempt to do any geometry that only relies on the module category.
- If $X$ is a commutative projective $\mathbb{k}$-scheme, for each $x \in X$, there is the skyscraper sheaf $\mathbb{k}(x) \in \text{coh}(X)$.
- So the simple objects of $\text{coh}(X)$ correspond to points of $X$.
- A point module is a graded right module $M$ such that $M$ is cyclic, generated in degree 0, and has $\dim_{\mathbb{k}} M_n = 1$ for all $n$.
- Fact: If $A$ is f.g. connected graded noetherian $\mathbb{k}$-algebra generated in degree 1, then the point modules are simple objects of $\text{qgr-}A$.
- “Points” of $\text{Proj}_{\text{NC}} A = \text{simple modules}$. 
Twisted homogeneous coordinate rings

• We can go from rings to schemes via Proj.
  
  \[ \mathcal{R} \leadsto \text{Proj} \, \mathcal{R} \]

• Can we go back? (Certainly not uniquely: \( \text{Proj} \, \mathcal{R}^{(d)} \cong \text{Proj} \, \mathcal{R} \).)
• \( X \) a projective scheme, \( \mathcal{L} \) a line bundle on \( X \)
• The homogeneous coordinate ring is

  \[ B(X, \mathcal{L}) = \mathbb{k} \oplus \bigoplus_{n \geq 1} H^0(X, \mathcal{L}^\otimes n). \]

**Theorem (Serre)**

Assume \( \mathcal{L} \) is ample. Then

1. \( B = B(X, \mathcal{L}) \) is a f.g. graded noetherian \( \mathbb{k} \)-algebra.
2. \( X = \text{Proj} \, B \) so qgr-\( B \equiv \text{coh}(X) \).
Twisted homogeneous coordinate rings

- $X$ a (commutative) projective scheme, $\mathcal{L}$ a line bundle, and $\sigma \in \text{Aut}(X)$. Define

$$\mathcal{L}^\sigma = \sigma^* \mathcal{L} \text{ and } \mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}.$$

- The twisted homogeneous coordinate ring is

$$B(X, \mathcal{L}, \sigma) = \mathbb{k} \oplus \bigoplus_{n \geq 1} H^0(X, \mathcal{L}_n).$$

- $B(X, \mathcal{L}, \sigma)$ is not necessarily commutative

**Theorem (Artin-Van den Bergh, 1990)**

Assume $\mathcal{L}$ is $\sigma$-ample. Then

1. $B = B(X, \mathcal{L}, \sigma)$ is a f.g. graded noetherian $\mathbb{k}$-algebra.
2. $\text{qgr-}B \equiv \text{coh}(X)$. 

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Noncommutative curves

- “The Hartshorne approach”
- $A$ a $\mathbb{k}$-algebra, $V \subseteq A$ a $\mathbb{k}$-subspace generating $A$ spanned by $\{1, a_1, \ldots, a_m\}$.
- $V_0 = \mathbb{k}$, $V_n$ spanned by monomials of length $n$ in the $a_i$.
- The Gelfand-Kirillov dimension of $A$.

$$\text{GKdim } A = \limsup_{n \to \infty} \log_n (\dim_k V_n)$$

- $\text{GKdim } \mathbb{k}[x_1, \ldots, x_m] = m$.
- So noncommutative projective curves should have $\text{GKdim } 2$. 
Noncommutative curves

Theorem (Artin-Stafford, 1995)

Let $A$ be a f.g. connected graded domain generated in degree 1 with $\text{GKdim}(A) = 2$. Then there exists a projective curve $X$, an automorphism $\sigma$ and invertible sheaf $\mathcal{L}$ such that up to a f.d vector space

$$A = B(X, \mathcal{L}, \sigma)$$

• As a corollary, $\text{qgr-}A \equiv \text{coh}(X)$.
• Or “noncommutative projective curves are commutative”.

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Noncommutative surfaces

• The “right” definition of a noncommutative polynomial ring?

**Definition**

A a f.g. connected graded \( k \)-algebra is **Artin-Schelter regular** if

1. \( \text{gldim} A = d < \infty \)
2. \( \text{GKdim} A < \infty \) and
3. \( \text{Ext}^i_A(\mathbb{k}, A) = \delta_{i,d} \mathbb{k} \).

• Behaves homologically like a commutative polynomial ring.
• \( \mathbb{k}[x_1, \ldots, x_m] \) is AS-regular of dimension \( m \).
• Noncommutative \( \mathbb{P}^2 \)s should be qgr-\( A \) for \( A \) AS-regular of dimension 3.
Noncommutative surfaces

Theorem (Artin-Tate-Van den Bergh, 1990)

Let $A$ be an AS-regular ring of dimension 3 generated in degree 1. Either

(a) $A = B(X, \mathcal{L}, \sigma)$ for $X = \mathbb{P}^2$ or $X = \mathbb{P}^1 \times \mathbb{P}^1$ or
(b) $A \twoheadrightarrow B(E, \mathcal{L}, \sigma)$ for an elliptic curve $E$.

• Or “noncommutative $\mathbb{P}^2$s are either commutative or contain a commutative curve”.

• Other noncommutative surfaces (noetherian connected graded domains of GKdim 3)?

• Noncommutative $\mathbb{P}^3$ (AS-regular of dimension 4)?
Z-graded rings

- Throughout, “graded” really meant \( \mathbb{N} \)-graded
- Way back to our first example:
  \[
  A_1 = \mathbb{k}\langle x, y \rangle/(xy - yx - 1)
  \]
- Ring of differential operators on \( \mathbb{k}[t] \)
  - \( x \leftrightarrow t \cdot \)
  - \( y \leftrightarrow d/dt \)
- \( A \) is \( \mathbb{Z} \)-graded by \( \deg x = 1, \deg y = -1 \)
- Simple noetherian domain of GK dim 2
- Exists an outer automorphism \( \omega \), reversing the grading
  \[
  \omega(x) = y \quad \omega(y) = -x
  \]
Sierra (2009)

- Sue Sierra, *Rings graded equivalent to the Weyl algebra*
- Classified all rings graded Morita equivalent to $A_1$
- Examined the graded module category $\text{gr}-A_1$:

  ![Diagram](image)

  - For each $\lambda \in \mathbb{k} \setminus \mathbb{Z}$, one simple module $M_\lambda$
  - For each $n \in \mathbb{Z}$, two simple modules, $X\langle n \rangle$ and $Y\langle n \rangle$

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- For each $n$, exists a nonsplit extension of $X\langle n \rangle$ by $Y\langle n \rangle$ and a nonsplit extension of $Y\langle n \rangle$ by $X\langle n \rangle$
Sierra (2009)

- Computed Pic(gr-A_1)
- Shift functor, S:

- Autoequivalence ω:
There exist $\iota_n$, autoequivalences of $\text{gr}-A_1$, permuting $X\langle n \rangle$ and $Y\langle n \rangle$ and fixing all other simple modules.

Also $\iota_i \iota_j = \iota_j \iota_i$ and $\iota_n^2 \cong \text{Id}_{\text{gr}-A}$

$\text{Pic}(\text{gr}-A_1) \cong (\mathbb{Z}/2\mathbb{Z})^{(\mathbb{Z})} \times D_\infty \cong \mathbb{Z}_{\text{fin}} \times D_\infty$. 

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• Paul Smith, *A quotient stack related to the Weyl algebra*

• Proves that $\text{Gr}-A_1 \equiv \text{Qcoh}_\chi$

• $\chi$ is a quotient stack “whose coarse moduli space is the affine line $\text{Spec} \ k[z]$, and whose stacky structure consists of stacky points $B\mathbb{Z}_2$ supported at each integer point”

• $\text{Gr}-A_1 \equiv \text{Gr}(C, \mathbb{Z}_{\text{fin}}) \equiv \text{Qcoh}_\chi$
Smith (2011)

- \( \mathbb{Z}_{\text{fin}} \) the group of finite subsets of \( \mathbb{Z} \), operation XOR
- Constructs a \( \mathbb{Z}_{\text{fin}} \) graded ring

\[
C := \bigoplus_{J \in \mathbb{Z}_{\text{fin}}} \hom(A_1, \iota_J A_1) \cong \frac{\mathbb{L}[x_n \mid n \in \mathbb{Z}]}{(x_n^2 + n = x_m^2 + m \mid m, n \in \mathbb{Z})}
\]

\[
\cong \mathbb{L}[z \mid \sqrt{z - n}]
\]

where \( \deg x_n = \{n\} \)
- \( C \) is commutative, integrally closed, non-noetherian PID

**Theorem (Smith)**

There is an equivalence of categories

\[
\text{Gr-}A_1 \equiv \text{Gr}(C, \mathbb{Z}_{\text{fin}}).
\]
Theorem (Artin-Stafford, 1995)
Let $A$ be a f.g. connected $\mathbb{N}$-graded domain generated in degree 1 with $\text{GKdim}(A) = 2$. Then there exists a projective curve $X$ such that
\[ \text{qgr-}A \equiv \text{coh}(X). \]

Theorem (Smith, 2011)
$A_1$ is a f.g. $\mathbb{Z}$-graded domain with $\text{GKdim}(A_1) = 2$. There exists a commutative ring $C$ and quotient stack $\chi$ such that
\[ \text{Gr-}A_1 \equiv \text{Gr}(C, \mathbb{Z}_\text{fin}) \equiv \text{Qcoh}(\chi). \]
Generalized Weyl algebras (GWAs)

- Introduced by V. Bavula
- $D$ a ring; $\sigma \in \text{Aut}(D)$; $a \in \mathbb{Z}(D)$
- The generalized Weyl algebra $D(\sigma, a)$ with base ring $D$

$$D(\sigma, a) = \frac{D\langle x, y \rangle}{\left( \begin{array}{c} xy = a \\ yx = \sigma(a) \\ dx = x\sigma(d), d \in D \\ dy = y\sigma^{-1}(d), d \in D \end{array} \right)}$$

**Theorem (Bell-Rogalski, 2015)**

Every *simple* $\mathbb{Z}$-graded domain of GKdim 2 is graded Morita equivalent to a GWA.
The main object

- $D = \mathbb{k}[z]; \quad \sigma(z) = z - 1; \quad a = f(z)$

$$A(f) \cong \frac{\mathbb{k}\langle x, y, z \rangle}{(xy = f(z), yx = f(z - 1), xz = (z + 1)x, yz = (z - 1)y)}$$

- Two roots $\alpha, \beta$ of $f(z)$ are congruent if $\alpha - \beta \in \mathbb{Z}$

Example (The first Weyl algebra)

Take $f(z) = z$

$$D(\sigma, a) = \frac{\mathbb{k}[z]\langle x, y \rangle}{(xy = z, yx = z - 1, zx = x(z - 1), zy = y(z + 1))} \cong \frac{\mathbb{k}\langle x, y \rangle}{(xy - yx - 1)} = A_1.$$
The main object

Properties of $A(f)$:

- Noetherian domain
- Krull dimension 1
- Simple if and only if no congruent roots
- $\text{gl.dim. } A(f) = \begin{cases} 
1, & f \text{ has neither multiple nor congruent roots} \\
2, & f \text{ has congruent roots but no multiple roots} \\
\infty, & f \text{ has a multiple root}
\end{cases}$
- We can give $A(f)$ a $\mathbb{Z}$ grading by $\deg x = 1$, $\deg y = -1$, $\deg z = 0$
Questions and strategy

For these generalized Weyl algebras $A(f)$:

- What does $\text{gr}-A(f)$ look like?
- What is $\text{Pic}(\text{gr}-A(f))$?
- Can we construct a commutative $\Gamma$-graded ring $C$ such that $\text{gr}(C, \Gamma) \equiv \text{gr}-A(f)$?

Strategy:

- First quadratic $f$
- Distinct, non-congruent roots
- Congruent roots
- Double root
Distinct, non-congruent roots

Let $f(z) = z(z + \alpha)$ for some $\alpha \in k \setminus \mathbb{Z}$.

$$A = A(f) \cong \mathbb{k}\langle x, y, z \rangle \cong \begin{pmatrix}
xy = z(z + \alpha) & yx = (z - 1)(z + \alpha - 1) \\
xz = (z + 1)x & yz = (z - 1)y
\end{pmatrix}$$

- We still have $A$ is simple, $\text{K.dim}(A) = \text{gl.dim}(A) = 1$
- Still exists an outer automorphism $\omega$ reversing the grading

$$\omega(x) = y \quad \omega(y) = x \quad \omega(z) = 1 + \alpha - z$$
Distinct, non-congruent roots

Graded simple modules:

- One simple graded module $M_\lambda$ for each $\lambda \in k \setminus (\mathbb{Z} \cup \mathbb{Z} + \alpha)$
- For each $n \in \mathbb{Z}$ two simple modules $X^0\langle n \rangle$ and $Y^0\langle n \rangle$
- For each $n \in \mathbb{Z}$ two simple modules $X^\alpha\langle n \rangle$ and $Y^\alpha\langle n \rangle$
- A nonsplit extension of $X^0\langle n \rangle$ by $Y^0\langle n \rangle$ and vice versa
- A nonsplit extension of $X^\alpha\langle n \rangle$ by $Y^\alpha\langle n \rangle$ and vice versa
Distinct, non-congruent roots

Theorem (W)

There exist numerically trivial autoequivalences, \( \iota_{(n,\emptyset)} \) permuting \( X^0\langle n \rangle \) and \( Y^0\langle n \rangle \) and fixing all other simple modules. Similarly, there exist \( \iota_{(\emptyset,n)} \) permuting \( X^\alpha\langle n \rangle \) and \( Y^\alpha\langle n \rangle \).

\[
\text{Pic}(\text{gr-}A) \cong (\mathbb{Z}/2\mathbb{Z})^{(\mathbb{Z})} \rtimes D_{\infty}
\]
Distinct, non-congruent roots

- Define a $\mathbb{Z}_{\text{fin}} \times \mathbb{Z}_{\text{fin}}$ graded ring $C$:

$$C = \mathbb{k}[a_n, b_n \mid n \in \mathbb{Z}]$$

modulo the relations

$$a_n^2 + n = a_m^2 + m \quad \text{and} \quad a_n^2 = b_n^2 + \alpha \quad \text{for all} \quad m, n \in \mathbb{Z}$$

with $\deg a_n = (\{n\}, \emptyset)$ and $\deg b_n = (\emptyset, \{n\})$

Theorem (W)

There is an equivalence of categories $\text{gr}(C, \mathbb{Z}_{\text{fin}} \times \mathbb{Z}_{\text{fin}}) \equiv \text{gr}-A$. 
Multiple root

Let \( f(z) = z^2 \).

\[
A = A(f) \cong \frac{\mathbb{k}\langle x, y, z \rangle}{\langle xy = z^2, yx = (z - 1)^2 \rangle}
\]

\[
\begin{aligned}
&x z = (z + 1)x,
y z = (z - 1)y,
\end{aligned}
\]

• \( \text{K.dim}(A) = 1 \)

• Exists an outer automorphism \( \omega \) reversing the grading

\[
\omega(x) = y \quad \omega(y) = x \quad \omega(z) = 1 - z
\]

• Now \( \text{gl.dim}(A) = \infty \)
Multiple root

The graded simple modules

- For each $\lambda \in k \setminus \mathbb{Z}$, $M_\lambda$
- For each $n \in \mathbb{Z}$, $X\langle n \rangle$ and $Y\langle n \rangle$

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- Nonsplit extensions of $X\langle n \rangle$ by $Y\langle n \rangle$ and vice versa. Also self-extensions of $X\langle n \rangle$ by $X\langle n \rangle$ and $Y\langle n \rangle$ and $Y\langle n \rangle$. 
Theorem (W)

There exist numerically trivial autoequivalences, $\iota_n$ permuting $X\langle n \rangle$ and $Y\langle n \rangle$ and fixing all other simple modules.

\[ \text{Pic}(\text{gr-}A) \cong (\mathbb{Z}/2\mathbb{Z})^{(\mathbb{Z})} \rtimes D_\infty \]
Multiple root

Define a $\mathbb{Z}_{\text{fin}}$ graded ring $B$:

$$B = \bigoplus_{J \in \mathbb{Z}_{\text{fin}}} \hom_A(A, \iota J A) \cong \frac{\mathbb{k}[z][a_n \mid n \in \mathbb{Z}]}{(b_n^2 = (z + n)^2 \mid n \in \mathbb{Z})}.$$ 

**Theorem (W)**

$B$ is a reduced, non-noetherian, non-domain of Kdim 1 with uncountably many prime ideals.

**Theorem (W)**

There is an equivalence of categories $\text{gr}(B, \mathbb{Z}_{\text{fin}}) \equiv \text{gr}-A$. 

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**Z-graded rings**

November 9, 2015
Two congruent roots

Let \( f(z) = z(z + m) \) for some \( m \in \mathbb{N} \).

\[
A = A(f) \cong \frac{\mathbb{K}\langle x, y, z \rangle}{\begin{pmatrix}
xy = z(z + m) & yx = (z - 1)(z + m - 1) \\
(xz = (z + 1)x & yz = (z - 1)y
\end{pmatrix}}
\]

What do we lose?

- \( A \) no longer simple
- There exist new finite-dimensional simple modules
- Now \( \text{gl.dim}(A) = 2 \)

What remains?

- Still have \( \text{K.dim}(A) = 1 \)
- Still exists an outer automorphism \( \omega \) reversing the grading

\[
\omega(x) = y \quad \omega(y) = x \quad \omega(z) = 1 + m - z
\]
Two congruent roots

The graded simple modules

- For each $\lambda \in k \setminus \mathbb{Z}, M_{\lambda}$
- For each $n \in \mathbb{Z}, X\langle n \rangle, Y\langle n \rangle, \text{ and } Z\langle n \rangle$

- $Z\langle n \rangle$ is finite dimensional
- proj. dim $Z\langle n \rangle = 2$

- Leads to "more" finite length modules than in previous cases
Two congruent roots

Theorem (W)
If $\mathcal{F}$ is an autoequivalence of gr-$A$ then there exists $a = \pm 1$ and $b \in \mathbb{Z}$ such that

$$\{\mathcal{F}(X\langle n \rangle), \mathcal{F}(Y\langle n \rangle)\} \cong \{X\langle an + b \rangle, Y\langle an + b \rangle\}$$

and

$$\mathcal{F}(Z\langle n \rangle) \cong Z\langle an + b \rangle.$$
Two congruent roots

- In this case, finite dimensional modules $\mathbb{Z}\langle n \rangle$.
- Consider the quotient category

$$qgr-A = \text{gr-}A/\text{fdim-}A$$

**Theorem (W)**
There is an equivalence of categories $\text{gr}(B, \mathbb{Z}_{\text{fin}}) \equiv qgr-A$. 
The upshot

In all cases,

- There exist numerically trivial autoequivalences permuting $X$ and $Y$ and fixing all other simples.
- $\text{Pic}(\text{gr-}A(f)) \cong (\mathbb{Z}_2)^\mathbb{Z} \rtimes D_\infty$.
- There exists a $\mathbb{Z}_{\text{fin}}$-graded commutative ring $R$ such that
  \[ \text{qgr-}A(f) \equiv \text{gr}(R, \mathbb{Z}_{\text{fin}}). \]
Questions

- $\mathbb{Z}_{\text{fin}}$-grading on $B$ gives an action of $\text{Spec } k\mathbb{Z}_{\text{fin}}$ on $\text{Spec } B$

$$\chi = \left[ \frac{\text{Spec } B}{\text{Spec } k\mathbb{Z}_{\text{fin}}} \right]$$

- What are the properties of $\chi$?
- Other $\mathbb{Z}$-graded domains of GK dimension 2?
- GWAs defined by non-quadratic $f$? Other base rings $D$?
  - $U(sl(2))$
  - Quantum Weyl algebra
  - Simple $\mathbb{Z}$-graded domains
Questions

Theorem (Smith, 2011)

$A_1$ a $\mathbb{Z}$-graded domain of GK dim 2. Stack $\chi$ such that

$$\text{Gr-}A_1 \equiv \text{Gr}(C, \mathbb{Z}_{\text{fin}}) \equiv \text{Qcoh}(\chi).$$

• For $A$ a $\mathbb{Z}$-graded domain of GK dim 2, is there always a stack?
• Construction of $B$: $\Gamma \subseteq \text{Pic}(\text{gr-}A)$

$$B = \bigoplus_{\mathcal{F} \in \Gamma} \text{Hom}_{\text{qgr-}A}(A, \mathcal{F}A)$$

• Take $\Gamma = \left\langle S \right\rangle = \mathbb{Z}$ then $B = A$.
• Opposite view: $\mathcal{O}$ a quasicoherent sheaf on $\chi$:

$$\bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\text{Qcoh}(\chi)}(\mathcal{O}, S^n\mathcal{O})$$

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