The graded module category of a generalized Weyl algebra
Final Defense

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Overview

1. Graded rings and things
2. Noncommutative is not commutative
3. Noncommutative projective schemes
4. $\mathbb{Z}$-graded rings
   - The first Weyl algebra
   - Generalized Weyl algebras
5. Future direction
Graded $\mathbb{k}$-algebras

- Throughout, $\mathbb{k}$ an algebraically closed field, $\text{char } \mathbb{k} = 0$.
- A $\mathbb{k}$-algebra is a ring $A$ with 1 which is a $\mathbb{k}$-vector space such that

$$\lambda(ab) = (\lambda a)b = a(\lambda b) \text{ for } \lambda \in \mathbb{k} \text{ and } a, b \in A.$$ 

**Examples**

- The polynomial ring, $\mathbb{k}[x]$
- The ring of $n \times n$ matrices, $M_n(\mathbb{k})$
- The complex numbers, $\mathbb{C} = \mathbb{R} \cdot 1 + \mathbb{R} \cdot i$
Graded $k$-algebras

- $\Gamma$ an abelian (semi)group
- A $\Gamma$-graded $k$-algebra $A$ has $k$-space decomposition

$$A = \bigoplus_{\gamma \in \Gamma} A_\gamma$$

such that $A_\gamma A_\delta \subseteq A_{\gamma+\delta}$.

- If $a \in A_\gamma$, $\deg a = \gamma$ and $a$ is homogeneous of degree $\gamma$. 
Graded \( k \)-algebras

Examples

- Any ring \( A \) with trivial grading
- \( k[x, y] \) graded by \( \mathbb{N} \)
- \( k[x, x^{-1}] \) graded by \( \mathbb{Z} \)
- If you like physics: superalgebras graded by \( \mathbb{Z}/2\mathbb{Z} \)
- Or combinatorics: symmetric and alternating polynomials graded by \( \mathbb{Z}/2\mathbb{Z} \)

Remark

Usually, “graded ring” means \( \mathbb{N} \)-graded.
Graded modules and morphisms

- A graded right $A$-module $M$:

$$M = \bigoplus_{i \in \mathbb{Z}} M_i$$

such that $M_i \cdot A_j \subseteq M_{i+j}$.

- Finitely generated graded right $A$-modules with graded module homomorphisms form a category $\text{gr-}A$.

Example

$\mathbb{N}$-graded ring $\mathbb{k}[x, y]$

- Homogeneous ideals $(x), (x, x + y), (x^2 - xy)$ are graded submodules

- Nonhomogeneous ideals $(x + 3), (x^2 + y)$ are not
Graded modules and morphisms

- Idea: Study $A$ by studying its module category, $\text{mod-} A$ (analogous to studying a group by its representations).
- If $\text{mod-} A \equiv \text{mod-} B$, we say $A$ and $B$ are Morita equivalent.

Remarks

- For commutative rings, Morita equivalent $\Leftrightarrow$ isomorphic
- $M_n(A)$ is Morita equivalent to $A$
- Morita invariant properties of $A$: simple, semisimple, noetherian, artinian, hereditary, etc.
Proj of a graded $\mathbb{k}$-algebra

- $A$ a commutative $\mathbb{N}$-graded $\mathbb{k}$-algebra
- The scheme $\text{Proj } A$
- As a set: homogeneous primes ideals of $A$ not containing $A_{\geq 0}$
- Topological space: the Zariski topology with closed sets

$$V(\alpha) = \{ p \in \text{Proj } A \mid \alpha \subseteq p \}$$

for homogeneous ideals $\alpha$ of $A$

- Structure sheaf: localize at homogeneous prime ideals
**Proj of a graded \( k \)-algebra**

**Interplay between algebra and geometry**

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<th>Geometry</th>
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<td>space ( \text{Proj} \ A )</td>
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<td>homogeneous prime ideals</td>
<td>points</td>
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<tr>
<td>( I \subseteq J )</td>
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<td>factor rings</td>
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<td>homogeneous elements of ( A )</td>
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</table>
Noncommutative is not commutative

Noncommutative rings are ubiquitous.

Examples

- Ring $A$, nonabelian group $G$, the group algebra $A[G]$
- $M_n(A)$, $n \times n$ matrices
- Differential operators on $k[t]$, generated by $t \cdot$ and $d/dt$ is isomorphic to the first Weyl algebra

\[ A_1 = k\langle x, y \rangle/(xy - yx - 1). \]

If you like physics, position and momentum operators don’t commute in quantum mechanics.
Noncommutative is not commutative

- Localization is different.
- Given $A$ commutative and $S \subset A$ multiplicatively closed,
  \[ a_1 s_1^{-1} a_2 s_2^{-1} = a_1 a_2 s_1^{-1} s_2^{-1} \]
- If $A$ noncommutative, can only form $AS^{-1}$ if $S$ is an Ore set.

**Definition**

$S$ is an Ore set if for any $a \in A$, $s \in S$

\[ sA \cap aS \neq \emptyset. \]
Noncommutative is not commutative

Even worse, not enough (prime) ideals.

The Weyl algebra

\[ \mathbb{k}\langle x, y \rangle/(xy - yx - 1) \]

is a noncommutative analogue of \( \mathbb{k}[x, y] \) but is **simple**.

The quantum polynomial ring

The \( \mathbb{N} \)-graded ring

\[ \mathbb{k}\langle x, y \rangle/(xy - qyx) \]

is a "noncommutative \( \mathbb{P}^1 \)" but for \( q^n \neq 1 \) has only three homogeneous ideals (namely \( (x) \), \( (y) \), and \( (x, y) \)).
Sheaves to the rescue

• The Beatles (paraphrased):

  “All you need is sheaves.”

• Idea: You can reconstruct the space from the sheaves.

Theorem (Rosenberg, Gabriel, Gabber, Brandenburg)
Let \( X, Y \) be quasi-separated schemes. If \( \text{qcoh}(X) \equiv \text{qcoh}(Y) \) then \( X \) and \( Y \) are isomorphic.
Sheaves to the rescue

• All you need is modules

**Theorem**

Let $X = \text{Proj} A$ for a commutative, f.g. $\mathbb{k}$-algebra $A$ generated in degree 1.

(1) Every coherent sheaf on $X$ is isomorphic to $\tilde{M}$ for some f.g. graded $A$-module $M$.

(2) $\tilde{M} \cong \tilde{N}$ as sheaves if and only if there is an isomorphism $M_{\geq n} \cong N_{\geq n}$.

• Let $\text{gr-}A$ be the category of f.g. $A$-modules. Take the quotient category

\[ \text{qgr-}A = \text{gr-}A/\text{fdim-}A \]

• The above says $\text{qgr-}A \cong \text{coh} (\text{Proj} A)$.  

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A (not necessarily commutative) connected graded $k$-algebra $A$ is

$$A = k \oplus A_1 \oplus A_2 \oplus \cdots$$

such that $\dim_k A_i < \infty$ and $A$ is a f.g. $k$-algebra.

**Definition (Artin-Zhang, 1994)**

The noncommutative projective scheme $\text{Proj}_{\text{NC}} A$ is the triple

$$(\text{qgr-}A, A, S)$$

where $A$ is the distinguished object and $S$ is the shift functor.
Noncommutative curves

- “The Hartshorne approach”
- A a \( k \)-algebra, \( V \subseteq A \) a \( k \)-subspace generating \( A \) spanned by \( \{1, a_1, \ldots, a_m\} \).
- \( V_0 = k, V_n \) spanned by monomials of length \( \leq n \) in the \( a_i \).
- The Gelfand-Kirillov dimension of \( A \)

\[
\text{GKdim } A = \limsup_{n \to \infty} \log_n (\text{dim}_k V_n)
\]

- \( \text{GKdim } k[x_1, \ldots, x_m] = m. \)
- So noncommutative projective curves should have \( \text{GKdim} 2. \)
Theorem (Artin-Stafford, 1995)

Let $A$ be a f.g. connected graded domain generated in degree 1 with $\text{GKdim}(A) = 2$. Then there exists a projective curve $X$ such that

$$qgr-A \equiv \text{coh}(X)$$

Or “noncommutative projective curves are commutative”.
Noncommutative surfaces

• The “right” definition of a noncommutative polynomial ring?

**Definition**

A a f.g. connected graded \( k \)-algebra is **Artin-Schelter regular** if

1. \( \text{gl.dim} \, A = d < \infty \)
2. \( \text{GKdim} \, A < \infty \) and
3. \( \text{Ext}^i_A (k, A) = \delta_{i,d} k. \)

• Interesting invariant theory for these rings.
• Behaves homologically like a commutative polynomial ring.
• \( \mathbb{k}[x_1, \ldots, x_m] \) is AS regular of dimension \( m \).
• Noncommutative \( \mathbb{P}^2 \)s should be \( \text{qgr-A} \) for \( A \) AS-regular of dimension 3.
Noncommutative surfaces

Theorem (Artin-Tate-Van den Bergh, 1990)

Let $A$ be an AS regular ring of dimension 3 generated in degree 1. Either

(a) $\text{qgr-}A \equiv \text{coh}(X)$ for $X = \mathbb{P}^2$ or $X = \mathbb{P}^1 \times \mathbb{P}^1$, or
(b) $A \twoheadrightarrow B$ and $\text{qgr-}B \equiv \text{coh}(E)$ for an elliptic curve $E$.

• Or “noncommutative $\mathbb{P}^2$s are either commutative or contain a commutative curve”.

• Other noncommutative surfaces (noetherian connected graded domains of GKdim 3)?

• Noncommutative $\mathbb{P}^3$ (AS-regular of dimension 4)?
The first Weyl algebra

• Throughout, “graded” really meant $\mathbb{N}$-graded
• Recall the first Weyl algebra:

$$A_1 = \mathbb{k}\langle x, y \rangle/(xy - yx - 1)$$

• Simple noetherian domain of GK dim 2
• $A_1$ is not $\mathbb{N}$-graded
The first Weyl algebra

\[ A_1 = \mathbb{k}\langle x, y \rangle / (xy - yx - 1) \]

- Ring of differential operators on \( \mathbb{k}[t] \)
  - \( x \leftrightarrow t \)
  - \( y \leftrightarrow d/dt \)
- \( A \) is \( \mathbb{Z} \)-graded by \( \deg x = 1, \deg y = -1 \)
- \( (A_1)_0 \cong \mathbb{k}[xy] \neq \mathbb{k} \)
- Exists an outer automorphism \( \omega \), reversing the grading
  \[ \omega(x) = y \quad \omega(y) = -x \]
Sierra (2009)

• Sue Sierra, *Rings graded equivalent to the Weyl algebra*
• Classified all rings graded Morita equivalent to $A_1$
• Examined the graded module category $\text{gr-}A_1$:

```
-3  -2  -1  0   1  2  3
```

• For each $\lambda \in k \setminus \mathbb{Z}$, one simple module $M_\lambda$
• For each $n \in \mathbb{Z}$, two simple modules, $X\langle n \rangle$ and $Y\langle n \rangle$

<table>
<thead>
<tr>
<th></th>
<th>$X$</th>
<th>$Y$</th>
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• For each $n$, exists a nonsplit extension of $X\langle n \rangle$ by $Y\langle n \rangle$ and a nonsplit extension of $Y\langle n \rangle$ by $X\langle n \rangle$
Sierra (2009)

- Computed Pic\((\text{gr-}A_1)\), the Picard group (group of autoequivalences) of gr-\(A_1\)
- Shift functor, \(S:\)

![Diagram]

- Autoequivalence \(\omega:\)

![Diagram]
Sierra (2009)

Theorem (Sierra)

There exist $\iota_n$, autoequivalences of $\text{gr-}A_1$, permuting $X\langle n \rangle$ and $Y\langle n \rangle$ and fixing all other simple modules.

Also $\iota_i \iota_j = \iota_j \iota_i$ and $\iota_n^2 \cong \text{Id}_{\text{gr-}A}$

Theorem (Sierra)

$\text{Pic}(\text{gr-}A_1) \cong (\mathbb{Z}/2\mathbb{Z})^{(\mathbb{Z})} \rtimes D_\infty \cong \mathbb{Z}_{\text{fin}} \rtimes D_\infty$. 

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• Paul Smith, *A quotient stack related to the Weyl algebra*

  \[ \text{Gr-} A_1 \equiv \text{Qcoh}(\chi) \]

  \[ \chi \text{ is a quotient stack “whose coarse moduli space is the affine line } \text{Spec } k[z], \text{ and whose stacky structure consists of stacky points } B\mathbb{Z}_2 \text{ supported at each integer point”} \]

  \[ \text{Gr-} A_1 \equiv \text{Gr}(C, \mathbb{Z}_{\text{fin}}) \equiv \text{Qcoh}(\chi) \]
Smith (2011)

- $\mathbb{Z}_{\text{fin}}$ the group of finite subsets of $\mathbb{Z}$, operation XOR
- Constructs a $\mathbb{Z}_{\text{fin}}$-graded ring

$$C := \bigoplus_{J \in \mathbb{Z}_{\text{fin}}} \text{hom}(A_1, \iota_J A_1) \cong k[x_n \mid n \in \mathbb{Z}] \frac{(x_n^2 + n = x_m^2 + m \mid m, n \in \mathbb{Z})}{(x_n^2 + n = x_m^2 + m \mid m, n \in \mathbb{Z})}$$

where $\deg x_n = \{n\}$

- $C$ is commutative, integrally closed, non-noetherian PID

Theorem (Smith)

There is an equivalence of categories

$$\text{Gr-}A_1 \equiv \text{Gr}(C, \mathbb{Z}_{\text{fin}}).$$
Theorem (Artin-Stafford, 1995)
Let $A$ be a f.g. connected $\mathbb{N}$-graded domain generated in degree 1 with $\text{GKdim}(A) = 2$. Then there exists a projective curve $X$ such that
\[ \text{qgr-}A \equiv \text{coh}(X). \]

Theorem (Smith, 2011)
$A_1$ is a f.g. $\mathbb{Z}$-graded domain with $\text{GKdim}(A_1) = 2$. There exists a commutative ring $C$ and quotient stack $\chi$ such that
\[ \text{gr-}A_1 \equiv \text{gr}(C, \mathbb{Z}_{\text{fin}}) \equiv \text{coh}(\chi). \]
Generalized Weyl algebras (GWAs)

• Introduced by V. Bavula
• *Generalized Weyl algebras and their representations* (1993)
• $D$ a ring; $\sigma \in \text{Aut}(D)$; $a \in \mathbb{Z}(D)$
• The generalized Weyl algebra $D(\sigma, a)$ with base ring $D$

\[
D(\sigma, a) = \frac{D\langle x, y \rangle}{\left(\begin{array}{c}
xy = a \\
yx = \sigma(a) \\
dx = x\sigma(d), d \in D \\
dy = y\sigma^{-1}(d), d \in D
\end{array}\right)}
\]

**Theorem (Bell-Rogalski, 2015)**

Every *simple* $\mathbb{Z}$-graded domain of GKdim 2 is graded Morita equivalent to a GWA.
Generalized Weyl algebras

- For us, $D = \mathbb{k}[z]$; $\sigma(z) = z - 1$; $a = f(z)$

$$A(f) \cong \frac{\mathbb{k}\langle x, y, z \rangle}{\left(\begin{array}{ccc} xy = f(z) & yx = f(z - 1) \\ xz = (z + 1)x & yz = (z - 1)y \end{array}\right)}$$

- Two roots $\alpha, \beta$ of $f(z)$ are congruent if $\alpha - \beta \in \mathbb{Z}$

Example (The first Weyl algebra)

Take $f(z) = z$

$$D(\sigma, a) = \frac{\mathbb{k}[z]\langle x, y \rangle}{\left(\begin{array}{ccc} xy = z & yx = z - 1 \\ zx = x(z - 1) & zy = y(z + 1) \end{array}\right)} \cong \frac{\mathbb{k}\langle x, y \rangle}{(xy - yx - 1)} = A_1.$$
Generalized Weyl algebras

Properties of $A(f)$:

- Noetherian domain
- Krull dimension 1
- Simple if and only if no congruent roots
- $\text{gl.dim } A(f) = \begin{cases} 1, & f \text{ has neither multiple nor congruent roots} \\ 2, & f \text{ has congruent roots but no multiple roots} \\ \infty, & f \text{ has a multiple root} \end{cases}$
- We can give $A(f)$ a $\mathbb{Z}$ grading by $\deg x = 1$, $\deg y = -1$, $\deg z = 0$
Questions

Focus on quadratic $f(z) = z(z + \alpha)$. For these GWAs $A(f)$:

- What does $\text{gr-}A(f)$ look like?
- What is $\text{Pic}(\text{gr-}A(f))$?
- Can we construct a commutative $\Gamma$-graded ring $C$ such that $\text{gr}(C, \Gamma) \equiv \text{gr-}A(f)$?
Results

If $\alpha \in k \setminus \mathbb{Z}$ (distinct roots):

If $\alpha = 0$ (double root):

If $\alpha \in \mathbb{N}^+$ (congruent roots):
Results

Theorem(s) (W)

In all cases, there exist numerically trivial autoequivalences $\iota_n$ permuting "$X\langle n\rangle$" and "$Y\langle n\rangle$" and fixing all other simple modules.

Corollary (W)

For quadratic $f$, $\text{Pic}(\text{gr}-A(f)) \cong \mathbb{Z}_{\text{fin}} \rtimes D_\infty$. 
Results

Let $\alpha \in k \setminus \mathbb{Z}$. Define a $\mathbb{Z}_{\text{fin}} \times \mathbb{Z}_{\text{fin}}$ graded ring $C$:

$$C = \bigoplus_{(J,J') \in \mathbb{Z}_{\text{fin}} \times \mathbb{Z}_{\text{fin}}} \text{hom}_A (A, \iota_{(J,J')}A)$$

with $\deg a_n = (\{n\}, \emptyset)$ and $\deg b_n = (\emptyset, \{n\})$

Theorem (W)

There is an equivalence of categories $\text{gr}(C, \mathbb{Z}_{\text{fin}} \times \mathbb{Z}_{\text{fin}}) \cong \text{gr-A}$. 
Results

Let $\alpha = 0$. Define a $\mathbb{Z}_{\text{fin}}$-graded ring $B$:

$$B = \bigoplus_{J \in \mathbb{Z}_{\text{fin}}} \text{hom}_A(A, \iota_J A) \cong \frac{k[z][b_n \mid n \in \mathbb{Z}]}{(b_n^2 = (z + n)^2 \mid n \in \mathbb{Z})}.$$ 

Theorem (W)

$B$ is a reduced, non-noetherian, non-domain of Kdim 1 with uncountably many prime ideals.

Theorem (W)

There is an equivalence of categories $\text{gr}(B, \mathbb{Z}_{\text{fin}}) \equiv \text{gr}-A$. 

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Results

- In congruent root ($\alpha \in \mathbb{N}^+$), we have finite-dimensional modules.
- Consider the quotient category
  \[ \text{qgr-}A = \text{gr-}A / \text{fdim-}A. \]

Theorem (W)
There is an equivalence of categories $\text{gr}(B, \mathbb{Z}_{\text{fin}}) \equiv \text{qgr-}A$. 
The upshot

In all cases,

• There exist numerically trivial autoequivalences permuting $X$ and $Y$ and fixing all other simples.

• $\text{Pic}(\text{gr-}A(f)) \cong \mathbb{Z}_{\text{fin}} \rtimes D_{\infty}$.

• There exists a $\mathbb{Z}_{\text{fin}}$-graded commutative ring $R$ such that

\[ \text{qgr-}A(f) \equiv \text{gr}(R, \mathbb{Z}_{\text{fin}}). \]
Questions

• $\mathbb{Z}_{\text{fin}}$-grading on $B$ gives an action of $\text{Spec } k\mathbb{Z}_{\text{fin}}$ on $\text{Spec } B$

$$\chi = \begin{bmatrix} \text{Spec } B \\ \text{Spec } k\mathbb{Z}_{\text{fin}} \end{bmatrix}$$

• Other $\mathbb{Z}$-graded domains of GK dimension 2? Other GWAs?
  • $U(\mathfrak{sl}(2))$, Down-up algebras, quantum Weyl algebra, Simple $\mathbb{Z}$-graded domains

• Construction of $B$: $\{F_\gamma \mid \gamma \in \Gamma\} \subseteq \text{Pic}(\text{gr-}A)$

$$B = \bigoplus_{\gamma \in \Gamma} \text{Hom}_{\text{qgr-}A}(A, F_\gamma A)$$

• Opposite view: $\mathcal{O}$ a quasicoherent sheaf on $\chi$:

$$\bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\text{Qcoh}(\chi)}(\mathcal{O}, S^n \mathcal{O})$$
Thank you!