The category of graded modules of a generalized Weyl algebra

Joint Mathematics Meetings, Seattle, WA

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Overview

1 Known
   Graded modules over the first Weyl algebra

2 Know
   Graded modules over generalized Weyl algebras

3 Now
   Future direction
The Weyl algebra

• Throughout, $\mathbb{k} = \overline{\mathbb{k}}$, $\text{char}(\mathbb{k}) = 0$
• A noncommutative $\mathbb{k}$-algebra, the first Weyl algebra
  \[ A_1 = \mathbb{k}\langle x, y \rangle / (xy - yx - 1) \]
• $A_1$ is $\mathbb{Z}$-graded by $\deg x = 1$, $\deg y = -1$
• Exists an outer automorphism $\omega$, reversing the grading
  \[ \omega(x) = y \quad \omega(y) = -x \]
Sierra (2009)

• Sue Sierra, *Rings graded equivalent to the Weyl algebra*
• Examined the graded module category $\text{gr-}A_1$:

- For each $\lambda \in \mathbb{k} \setminus \mathbb{Z}$, one simple module $M_\lambda$
- For each $n \in \mathbb{Z}$, two simple modules, $X\langle n \rangle$ and $Y\langle n \rangle$
• Computed $\text{Pic}(\text{gr-}A_1)$ (autoequivalences of $\text{gr-}A_1$)

• Shift functor, $S$:

• Autoequivalence $\omega$:
There exist $\iota_n$, autoequivalences of gr-$A_1$, permuting $X\langle n \rangle$ and $Y\langle n \rangle$ and fixing all other simple modules.
Smith (2011)

• Paul Smith, *A quotient stack related to the Weyl algebra*

\[ \text{Gr-}A_1 \equiv \text{Qcoh} \chi \]

• Proves that \( \text{Gr-}A_1 \equiv \text{Qcoh} \chi \)

• \( \chi \) is a quotient stack “whose coarse moduli space is the affine line \( \text{Spec } k[z] \), and whose stacky structure consists of stacky points \( B\mathbb{Z}_2 \) supported at each integer point”

• \( \text{Gr-}A_1 \equiv \text{Gr}(C, \mathbb{Z}_{\text{fin}}) \equiv \text{Qcoh} \chi \)
• $\mathbb{Z}_{\text{fin}}$ the group of finite subsets of $\mathbb{Z}$, operation XOR
• Constructs a $\mathbb{Z}_{\text{fin}}$ graded ring

$$C := \bigoplus_{J \in \mathbb{Z}_{\text{fin}}} \text{hom}(A, \iota_J A) \cong \mathbb{k}[z] [\sqrt{z - n} \mid n \in \mathbb{Z}]$$

where $\deg \sqrt{z - n} = \{n\}$
• $C$ is commutative, integrally closed, non-noetherian PID

**Theorem (Smith, 2011)**

There is an equivalence of categories

$$\text{Gr}-A_1 \equiv \text{Gr}(C, \mathbb{Z}_{\text{fin}}).$$
**Z-graded geometry?**

<table>
<thead>
<tr>
<th>Theorem (Artin-Stafford, 1995)</th>
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Generalized Weyl algebras (GWAs)

- For \( f \in k[z] \), the generalized Weyl algebra defined by \( f \)

\[
A(f) \cong \frac{k\langle x, y, z \rangle}{\left( \begin{array}{c} xy = f(z) \\
yx = f(z - 1) \\
xz = (z + 1)x \\
yz = (z - 1)y \end{array} \right)}
\]

- \( \mathbb{Z} \)-graded by \( \deg x = 1, \deg y = -1, \deg z = 0 \)

**Example (The first Weyl algebra)**

Take \( f = z \)

\[
A(f) = \frac{k\langle x, y, z \rangle}{\left( \begin{array}{c} xy = z \\
yx = z - 1 \\
zx = x(z - 1) \\
zy = y(z + 1) \end{array} \right)} \cong \frac{k\langle x, y \rangle}{(xy - yx - 1)} = A_1.
\]
GWAs and graded modules

Consider quadratic $f$. WLOG, three cases (depending on the roots of $f$)

- $f = z(z + \alpha), \alpha \notin \mathbb{Z}$

$$\begin{align*}
\alpha - 2 & & \alpha - 1 & & \alpha & & \alpha + 1 \\
-2 & & -1 & & 0 & & 1 & & 2
\end{align*}$$

- $f = z^2$

$$\begin{align*}
-3 & & -2 & & -1 & & 0 & & 1 & & 2 & & 3
\end{align*}$$

- $f = z(z + \alpha), \alpha \in \mathbb{N}$

$$\begin{align*}
-3 & & -2 & & -1 & & 0 & & 1 & & 2 & & 3
\end{align*}$$
GWAs and graded modules

**Theorem (W)**

For all quadratic $f$, there exist autoequivalences permuting $X\langle n \rangle$ and $Y\langle n \rangle$ and fixing all other simple modules.

**Theorem (W)**

For all quadratic $f$, there exists a $\mathbb{Z}_{\text{fin}}$-graded commutative ring $B(f)$ such that

$$qgr-A(f) \equiv \text{gr}(B(f), \mathbb{Z}_{\text{fin}}).$$
GWAs and graded modules

If \( f = z(z + \alpha) \) for \( \alpha \in \mathbb{Z} \), then

\[
B(f) = \bigoplus_{J \in \mathbb{Z}_{\text{fin}}} \text{hom}_A(A, \nu_J A) \cong \frac{\mathbb{k}[z][a_n \mid n \in \mathbb{Z}]}{(a_n^2 = (z + n)^2 \mid n \in \mathbb{Z})}.
\]

**Theorem (W)**

\( B(f) \) is a reduced, non-noetherian, non-domain of Kdim 1 with uncountably many prime ideals.

If \( f = z(z + \alpha) \) for \( \alpha \notin \mathbb{Z} \), then

\[
B(f) \cong \frac{\mathbb{k}[z][a_n, b_n \mid n \in \mathbb{Z}]}{(a_n^2 = z + n, b_n^2 = z + n + \alpha \mid n \in \mathbb{Z})}.
\]
Questions

• $\mathbb{Z}_{\text{fin}}$-grading on $B$ gives an action of $\text{Spec } \mathbb{k}\mathbb{Z}_{\text{fin}}$ on $\text{Spec } B$

\[
\chi = \begin{bmatrix}
\text{Spec } B \\
\text{Spec } \mathbb{k}\mathbb{Z}_{\text{fin}}
\end{bmatrix}
\]

• What are the properties of $\chi$?
• Other $\mathbb{Z}$-graded domains of GK dimension 2?
• GWAs defined by non-quadratic $f$? Other base rings $D$?
  • $U(\mathfrak{sl}(2))$
  • Quantum Weyl algebra
  • Simple $\mathbb{Z}$-graded domains
Questions

• Construction of $B$: $\Gamma \subseteq \text{Pic} (\text{gr-A})$

$$B = \bigoplus_{\mathcal{F} \in \Gamma} \text{Hom}_{\text{qgr-A}}(A, \mathcal{F}A)$$

• Take $\Gamma = \langle S \rangle = \mathbb{Z}$ then $B = A$.

• Opposite view: $\mathcal{O}$ a quasicoherent sheaf on $\chi$:

$$\bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\text{Qcoh}(\chi)}(\mathcal{O}, S^n\mathcal{O})$$